Controlled SPDEs: Peng's Maximum Principle and Numerical Methods

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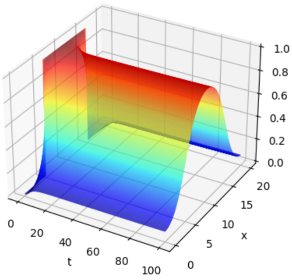
Example

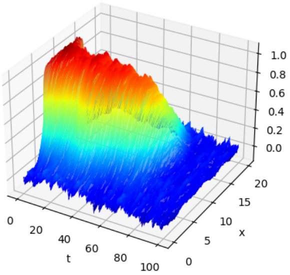
Consider the Nagumo equation

$$\begin{cases} \mathrm{d}u_t = \left[\Delta u_t + u_t(u_t - 1)\left(\frac{1}{2} - u_t\right)\right]\mathrm{d}t + \sigma \mathrm{d}W_t, & t \in [0,100] \\ u_0 = u \in L^2(0,20) \end{cases}$$

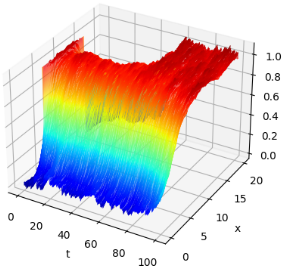
with Neumann boundary conditions and $u = 1_{[5,15]}$.



$\sigma = 0$



$\sigma = 0.05$



$\sigma = 0.05$

Control of the Stochastic Nagumo Equation

Introduce control $g : [0, 100] \times [0, 20] \times \Omega \to \mathbb{R}$

$$\begin{cases} \mathrm{d}u_t^g = \left[\Delta u_t^g + u_t^g(u_t^g - 1)\left(\frac{1}{2} - u_t^g\right) + g_t\right]\mathrm{d}t + \sigma \mathrm{d}W_t, \quad t \in [0, 100] \\ u_0^g = u \in L^2(0, 20) \end{cases} \qquad (\star)$$

and cost functional

$$J(g) = \mathbb{E}\left[\int_0^{100}\int_\Lambda \left(u_t^g(x) - u^{\mathrm{ref}}(t, x)\right)^2 + g^2(t, x)\mathrm{d}x\mathrm{d}t + \int_\Lambda \left(u_{100}^g(x) - u^T(x)\right)^2 \mathrm{d}x\right]$$

where u^{ref}, u^T are desired reference profiles.

Goal:

$$\boxed{\text{Minimize} \quad J \quad \text{subject to } (\star)}$$

Outline

Setting

Minimize

$$J(g) := \mathbb{E}\left[\int_0^T \int_\Lambda l(u_t^g(x), g_t)\mathrm{d}x\mathrm{d}t + \int_\Lambda m(u_T^g(x))\mathrm{d}x\right]$$

subject to

$$\begin{cases} \mathrm{d}u_t^g = [\Delta u_t^g + b(u_t^g, g_t)]\mathrm{d}t + \sigma(u_t^g, g_t)\mathrm{d}W_t, & t \in [0, T] \\ u_0^g = u_0 \in L^2(\Lambda), \end{cases}$$

where

- l, m Nemytskii operators of (at most) quadratic growth
- b, σ Nemytskii operators, Lipschitz
- $(W_t)_{t\geq 0}$ cylindrical Wiener process
- $\Lambda \subset \mathbb{R}^d$ bounded domain
- control domain $\mathcal{U}$ non-convex.

Pontryagin 1956: Controlled ODEs

Minimize

$$J(g) := \int_0^T l(u_t^g, g_t)\mathrm{d}t + m(u_T^g)$$

subject to

$$\begin{cases} \partial_t u_t^g = b(u_t^g, g_t), & t \in [0, T] \\ u_0^g = u_0 \in \mathbb{R}^n. \end{cases}$$

Let $(\bar{u}, \bar{g})$ be optimal. Then

$$\inf_{g \in \mathcal{U}} \mathcal{H}(t, \bar{u}_t, g) = \mathcal{H}(t, \bar{u}_t, \bar{g}_t),$$

for almost every $t \in [0, T]$, where Hamiltonian $\mathcal{H} : [0, T] \times \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}$

$$\mathcal{H}(t, u, g) := l(u, g) + \langle p_t, b(u, g) \rangle$$

where p_t is the adjoint state.

Peng 1990: Controlled SDEs

Minimize

$$J(g) := \mathbb{E}\left[\int_0^T l(u_t^g, g_t)\mathrm{d}t + m(u_T^g)\right]$$

subject to

$$\begin{cases} \mathrm{d}u_t^g = b(u_t^g, g_t)\mathrm{d}t + \sigma(u_t^g, g_t)\mathrm{d}W_t, \quad t \in [0, T] \\ u_0^g = u_0 \in \mathbb{R}^n. \end{cases}$$

Let $(\bar{u}, \bar{g})$ be optimal. Then

$$\inf_{g \in \mathcal{U}} \mathcal{G}(t, \bar{u}_t, g) = \mathcal{G}(t, \bar{u}_t, \bar{g}_t),$$

for almost all $(t, \omega) \in [0, T] \times \Omega$, where generalized Hamiltonian $\mathcal{G} : [0, T] \times \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}$

$$\mathcal{G}(t, u, g) := \mathcal{H}(t, u, g) + \frac{1}{2}\mathrm{tr}(\sigma(u, g)^* P_t \sigma(u, g)) + \mathrm{tr}\left(\sigma(u, g)^* \left[q_t - P_t \sigma(\bar{u}_t, \bar{g}_t)\right]\right)$$

where (p_t, q_t) first order adjoint state and $P_t \in \mathbb{R}^{n \times n}$ second order adjoint state.

Now: Controlled SPDEs

Minimize

$$J(g) := \mathbb{E}\left[\int_0^T \int_\Lambda l(u_t^g(x), g_t)\mathrm{d}x\mathrm{d}t + \int_\Lambda m(u_T^g(x))\mathrm{d}x\right]$$

subject to

$$\begin{cases} \mathrm{d}u_t^g = [\Delta u_t^g + b(u_t^g, g_t)]\mathrm{d}t + \sigma(u_t^g, g_t)\mathrm{d}W_t, & t \in [0, T] \\ u_0^g = u_0 \in L^2(\Lambda). \end{cases}$$

- Since Peng's result: Many generalizations to infinite-dimensional stochastic case by Du, Fuhrman, Frankowska, Guatteri, Hu, Li, Lü, Meng, Tang, Tessitore, Zhang, ...
- Major drawback in previous works: Restrictive assumptions on coefficients l and m, in particular excluding quadratic costs.
- Main obstacle: Characterization of second order adjoinst state $P_t \in L(H)$.

Spike Variation

Let $(\bar{u}, \bar{g})$ be optimal. Fix $\tau \in [0, T]$, $\varepsilon > 0$, $h \in \mathcal{U}$, and set

$$g_t^\varepsilon := \begin{cases} h, & \tau \le t \le \tau + \varepsilon, \\ \bar{g}_t & \text{otherwise.} \end{cases}$$

Then

$$0 \le J(g^\varepsilon) - J(\bar{g}) = \mathbb{E}\left[\int_0^T \int_\Lambda l(u_t^\varepsilon, g_t^\varepsilon) - l(\bar{u}_t, \bar{g}_t) \mathrm{d}x \mathrm{d}t + \int_\Lambda m(u_T^\varepsilon) - m(\bar{u}_T) \mathrm{d}x \right].$$

Taylor expansions: For terminal costs we have

$$\mathbb{E}\left[\int_\Lambda m(u_T^\varepsilon) - m(\bar{u}_T) \mathrm{d}x \right] \approx \mathbb{E}\left[\int_\Lambda m_u(\bar{u}_T) v_T^\varepsilon \mathrm{d}x \right],$$

where v^ε satisfies linearized state equation

$$\begin{cases} \mathrm{d}v_t^\varepsilon = \left[\Delta v_t^\varepsilon + b_u(\bar{u}_t, \bar{g}_t) v_t^\varepsilon + b(\bar{u}_t, g_t^\varepsilon) - b(\bar{u}_t, \bar{g}_t) \right] \mathrm{d}t \\ \qquad\qquad + \left[\sigma_u(\bar{u}_t, \bar{g}_t) v_t^\varepsilon + \sigma(\bar{u}_t, g_t^\varepsilon) - \sigma(\bar{u}_t, \bar{g}_t) \right] \mathrm{d}W_t \\ v_0^\varepsilon = 0. \end{cases}$$

Asymptotic of First Order Taylor Expansions

Applying Itô's formula to $\|v_t^\varepsilon\|^2_{L^2(\Lambda)}$ yields

$$\begin{aligned}\|v_T^\varepsilon\|^2_{L^2(\Lambda)} =& 2\int_0^T \langle \Delta v_s^\varepsilon + b_u(\bar{u}_s,\bar{g}_s)v_s^\varepsilon + b(\bar{u}_s,g_s^\varepsilon) - b(\bar{u}_s,\bar{g}_s), v_s^\varepsilon\rangle \mathrm{d}s\\ &+ 2\int_0^T \langle v_s^\varepsilon, (\sigma_u(\bar{u}_s,\bar{g}_s)v_s^\varepsilon + \sigma(\bar{u}_s,g_s^\varepsilon) - \sigma(\bar{u}_s,\bar{g}_s))\mathrm{d}W_s\rangle\\ &+ \int_0^T \|\sigma_u(\bar{u}_s,\bar{g}_s)v_s^\varepsilon + \sigma(\bar{u}_s,g_s^\varepsilon) - \sigma(\bar{u}_s,\bar{g}_s)\|^2_{L_2(\Xi,L^2(\Lambda))}\mathrm{d}s.\end{aligned}$$

Itô correction term

$$\begin{aligned}&\int_0^T \|\sigma_u(\bar{u}_s,\bar{g}_s)v_s^\varepsilon + \sigma(\bar{u}_s,g_s^\varepsilon) - \sigma(\bar{u}_s,\bar{g}_s)\|^2_{L_2(\Xi,L^2(\Lambda))}\mathrm{d}s\\ &\leq 2\int_0^T \|\sigma_u(\bar{u}_s,\bar{g}_s)v_s^\varepsilon\|^2_{L_2(\Xi,L^2(\Lambda))}\mathrm{d}s + 2\int_\tau^{\tau+\varepsilon} \|\sigma(\bar{u}_s,h) - \sigma(\bar{u}_s,\bar{g}_s)\|^2_{L_2(\Xi,L^2(\Lambda))}\mathrm{d}s.\end{aligned}$$

The last term is of order $\mathcal{O}(\varepsilon)$; we need $o(\varepsilon)$. $\rightsquigarrow$ need second order Taylor expansions!

Variational Equations

First order:

$$\begin{cases} \mathrm{d}v_t^\varepsilon = [\Delta v_t^\varepsilon + b_u(\bar{u}_t, \bar{g}_t)v_t^\varepsilon + b(\bar{u}_t, g_t^\varepsilon) - b(\bar{u}_t, \bar{g}_t)]\,\mathrm{d}t \\ \qquad\qquad + [\sigma_u(\bar{u}_t, \bar{g}_t)v_t^\varepsilon + \sigma(\bar{u}_t, g_t^\varepsilon) - \sigma(\bar{u}_t, \bar{g}_t)]\,\mathrm{d}W_t \\ v_0^\varepsilon = 0. \end{cases}$$

Second order:

$$\begin{cases} \mathrm{d}w_t^\varepsilon = \left[\Delta w_t^\varepsilon + b_u(\bar{u}_t, \bar{g}_t)w_t^\varepsilon + \frac{1}{2}b_{uu}(\bar{u}_t, \bar{g}_t)v_t^\varepsilon v_t^\varepsilon + (b_u(\bar{u}_t, g_t^\varepsilon) - b_u(\bar{u}_t, \bar{g}_t))v_t^\varepsilon\right]\mathrm{d}t \\ \qquad\qquad + \left[\sigma_u(\bar{u}_t, \bar{g}_t)w_t^\varepsilon + \frac{1}{2}\sigma_{uu}(\bar{u}_t, \bar{g}_t)v_t^\varepsilon v_t^\varepsilon + (\sigma_u(\bar{u}_t, g_t^\varepsilon) - \sigma_u(\bar{u}_t, \bar{g}_t))v_t^\varepsilon\right]\mathrm{d}W_t \\ w_0^\varepsilon = 0. \end{cases}$$

Lemma

It holds that

$$\sup_{t\in[0,T]} \mathbb{E}\left[\|v_t^\varepsilon\|_{L^2(\Lambda)}^2\right] \le C\varepsilon \qquad \textit{and} \qquad \sup_{t\in[0,T]} \mathbb{E}\left[\|w_t^\varepsilon\|_{L^2(\Lambda)}^2\right] \le C\varepsilon^2.$$

Second Order Expansion of Cost Functional

From

$$0 \leq J(g^{\varepsilon}) - J(\bar{g}) = \mathbb{E}\left[\int_0^T \int_\Lambda l(u_t^{\varepsilon}, g_t^{\varepsilon}) - l(\bar{u}_t, \bar{g}_t) \mathrm{d}x\mathrm{d}t + \int_\Lambda m(u_T^{\varepsilon}) - m(\bar{u}_T)\mathrm{d}x\right]$$

we derive:

Lemma

It holds

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T \int_\Lambda l_u(\bar{u}_t(x), \bar{g}_t)(v_t^{\varepsilon}(x) + w_t^{\varepsilon}(x)) + \frac{1}{2} l_{uu}(\bar{u}_t(x), \bar{g}_t) v_t^{\varepsilon}(x) v_t^{\varepsilon}(x)\mathrm{d}x\mathrm{d}t\right] \\
+&\mathbb{E}\left[\int_\Lambda m_u(\bar{u}_T(x))(v_T^{\varepsilon}(x) + w_T^{\varepsilon}(x)) + \frac{1}{2} m_{uu}(\bar{u}_T(x)) v_T^{\varepsilon}(x) v_T^{\varepsilon}(x)\mathrm{d}x\right] \\
+&\mathbb{E}\left[\int_0^T \int_\Lambda l(\bar{u}_t(x), g_t^{\varepsilon}) - l(\bar{u}_t(x), \bar{g}_t)\mathrm{d}x\mathrm{d}t\right] \geq o(\varepsilon).
\end{aligned}$$

Next step: Separate dependence on ε.

First Order Adjoint State

Consider SPDE

$$\begin{cases} \mathrm{d}v_t = [\Delta v_t + b_u(\bar{u}_t, \bar{g}_t)v_t + \varphi_t]\,\mathrm{d}t + [\sigma_u(\bar{u}_t, \bar{g}_t)v_t + \psi_t]\,\mathrm{d}W_t \\ v_0 = 0, \end{cases}$$

where $(\varphi, \psi) \in L^2([0,T] \times \Omega; L^2(\Lambda)) \times L^2([0,T] \times \Omega; L_2(\Xi, L^2(\Lambda)))$.
Construct linear functional

$$\mathcal{T}(\varphi, \psi) := \mathbb{E}\left[\int_0^T \int_\Lambda l_u(\bar{u}_t(x), \bar{g}_t)v_t(x)\mathrm{d}x\mathrm{d}t + \int_\Lambda m_u(\bar{u}_T(x))v_T(x)\mathrm{d}x\right].$$

By Riesz's representation theorem, there is a unique pair

$$(p, q) \in L^2([0,T] \times \Omega; L^2(\Lambda)) \times L^2([0,T] \times \Omega; L_2(\Xi, L^2(\Lambda)))$$

such that

$$\mathcal{T}(\varphi, \psi) = \mathbb{E}\left[\int_0^T \langle \varphi_t, p_t\rangle_{L^2(\Lambda)} + \langle \psi_t, q_t\rangle_{L_2(\Xi, L^2(\Lambda))}\mathrm{d}t\right]$$

for all $(\varphi, \psi) \in L^2([0,T] \times \Omega; L^2(\Lambda)) \times L^2([0,T] \times \Omega; L_2(\Xi, L^2(\Lambda)))$.

First Order Adjoint Equation

Adjoint state property:

$$\mathbb{E}\left[\int_0^T \langle v_t, l_u(\bar{u}_t, \bar{g}_t)\rangle \mathrm{d}t + \langle m_u(\bar{u}_T), v_T\rangle\right] = \mathbb{E}\left[\int_0^T \langle p_t, \varphi_t\rangle + \langle q_t, \psi_t\rangle \mathrm{d}t\right].$$

Applying Itô's product rule yields

$$\begin{aligned}\mathrm{d}\langle p_t, v_t\rangle_{L^2(\Lambda)} =& \langle p_t, \mathrm{d}v_t\rangle_{L^2(\Lambda)} + \langle v_t, \mathrm{d}p_t\rangle_{L^2(\Lambda)} + \mathrm{d}\langle p, v\rangle_t \\ \overset{!}{=}& \left[\langle p_t, \varphi_t\rangle_{L^2(\Lambda)} + \langle q_t, \psi_t\rangle_{L_2(\Xi, L^2(\Lambda))} - \langle v_t, l_u(\bar{u}_t, \bar{g}_t)\rangle_{L^2(\Lambda)}\right] \mathrm{d}t + dM_t.\end{aligned}$$

for some martingale $(M_t)_{t\geq 0}$. Thus,

$$\begin{cases} \mathrm{d}p_t = -\left[\Delta p_t + b_u(\bar{u}_t, \bar{g}_t)p_t + l_u(\bar{u}_t, \bar{g}_t) + \langle \sigma_u(\bar{u}_t, \bar{g}_t), q_t\rangle_{L_2(\Xi, \mathbb{R})}\right] \mathrm{d}t + q_t \mathrm{d}W_t \\ p_T = m_u(\bar{u}_T). \end{cases}$$

Unique variational solution (p, q), where

$$p \in L^2([0, T] \times \Omega; H_0^1(\Lambda)) \cap L^2(\Omega; C([0, T]; L^2(\Lambda)))$$

and

$$q \in L^2([0, T] \times \Omega; L_2(\Xi, L^2(\Lambda))).$$

Quadratic Terms

Follow same route for quadratic terms

$$\mathbb{E}\left[\int_0^T \int_\Lambda l_{uu}(\bar{u}_t(x),\bar{g}_t)v_t^\varepsilon(x)v_t^\varepsilon(x)\mathrm{d}x\mathrm{d}t + \int_\Lambda m_{uu}(\bar{u}_T(x))v_T^\varepsilon(x)v_T^\varepsilon(x)\mathrm{d}x\right].$$

Peng's idea in finite dimensions: Linearize using tensor product $v_t^\varepsilon \otimes v_t^\varepsilon$ and derive equation on

$$\mathbb{R}^n \otimes \mathbb{R}^n \cong \mathbb{R}^{n\times n}.$$

Infinite dimensional analogue:

$$H \otimes H \cong L_2(H).$$

Problem: Quadratic costs require duality analysis between $L_1(H)$ and $L(H)$.

Explicit Tensor Product

Instead, we use explicit representation

$$\begin{aligned} L^2(\Lambda) \otimes L^2(\Lambda) &\cong L^2(\Lambda^2) \\ v \otimes w &\leftrightsquigarrow ((x,y) \mapsto v(x)w(y)) \,. \end{aligned}$$

Rewrite quadratic terms as

$$\begin{aligned} &\mathbb{E}\left[\int_0^T \int_\Lambda l_{uu}(\bar{u}_t(x), \bar{g}_t) v_t^\varepsilon(x) v_t^\varepsilon(x) \mathrm{d}x\mathrm{d}t\right] \\ &= \mathbb{E}\left[\int_0^T \int_\Lambda l_{uu}(\bar{u}_t(x), \bar{g}_t) \delta(v_t^\varepsilon \otimes v_t^\varepsilon)(x) \mathrm{d}x\mathrm{d}t\right], \end{aligned}$$

where

$$\begin{aligned} \delta : H_0^1(\Lambda^2) &\to L^2(\Lambda) \\ V &\mapsto (x \mapsto V(x,x)). \end{aligned}$$

Second Order Adjoint Equation

Theorem (Stannat, W., SICON 2021)

The equation

$$\begin{cases} \mathrm{d}P_t(x,y) = -[\Delta P_t(x,y) + (b_u(\bar{u}_t(x),\bar{g}_t) + b_u(\bar{u}_t(y),\bar{g}_t))P_t(x,y) \\ \qquad\qquad +\langle \sigma_u(\bar{u}_t(x),\bar{g}_t), \sigma_u(\bar{u}_t(y),\bar{g}_t)\rangle_{L_2(\Xi,\mathbb{R})} P_t(x,y) \\ \qquad\qquad +\langle \sigma_u(\bar{u}_t(x),\bar{g}_t) + \sigma_u(\bar{u}_t(y),\bar{g}_t), Q_t(x,y)\rangle_{L_2(\Xi,\mathbb{R})} \\ \qquad\qquad +\delta^*(l_{uu}(\bar{u}_t(x),\bar{g}_t)) + \delta^*(b_{uu}(\bar{u}_t(x),\bar{g}_t)p_t(x)) \\ \qquad\qquad +\delta^*(\langle \sigma_{uu}(\bar{u}_t(x),\bar{g}_t), q_t\rangle_{L_2(\Xi,\mathbb{R})})]\mathrm{d}t + Q_t(x,y)\mathrm{d}W_t \\ P_T(x,y) = \delta^*(m_{uu}(\bar{u}_T(x))) \end{cases}$$

has a unique adapted solution (P,Q), *where*

$$P \in L^2([0,T]\times\Omega; L^2(\Lambda^2)) \cap L^2(\Omega; C([0,T]; H^{-1}(\Lambda^2))),$$

and

$$Q \in L^2([0,T]\times\Omega; L_2(\Xi; H^{-1}(\Lambda^2))).$$

Peng's Maximum Principle for SPDEs

Theorem (Stannat, W., SICON 2021)

Let $(\bar{u}, \bar{g})$ *be an optimal pair. Then there exist adapted processes*

$$(p, q) \in L^2([0, T] \times \Omega; H_0^1(\Lambda)) \times L^2([0, T] \times \Omega; L_2(\Xi, L^2(\Lambda)))$$

satisfying the first order adjoint equation and adapted processes

$$(P, Q) \in L^2([0, T] \times \Omega; L^2(\Lambda^2)) \times L^2([0, T] \times \Omega; L_2(\Xi, H^{-1}(\Lambda^2)))$$

satisfying the second order adjoint equation such that

$$\inf_{g \in \mathcal{U}} \mathcal{G}(t, \bar{u}_t, g) = \mathcal{G}(t, \bar{u}_t, \bar{g}_t),$$

for almost all $(t, \omega) \in [0, T] \times \Omega$*, where* $\mathcal{G} : [0, T] \times L^2(\Lambda) \times \mathcal{U} \to \mathbb{R}$

$$\begin{aligned}\mathcal{G}(t, u, g) := \int_\Lambda l(u(x), g)\mathrm{d}x + \langle p_t, b(u, g)\rangle_{L^2(\Lambda)} + tr\left(\sigma(u, g)^* q_t\right) \\ + \frac{1}{2} tr\left(\sigma(u, g)^* P_t \sigma(u, g)\right) - tr\left(\sigma(u, g)^* P_t \sigma(\bar{u}_t, \bar{g}_t)\right).\end{aligned}$$

Feedback Controls

Consider feedback control

$$G : [0, T] \times L^2(\Lambda) \to L^2(\Lambda)$$

and controlled SPDE

$$\begin{cases} du_t^G = [\Delta u_t^G + b(u_t^G) + G(t, u_t^G)]dt + \sigma dW_t, \ t \in [0, T] \\ u_0^G = u \in L^2(\Lambda). \end{cases}$$

Approximate by artificial neural network

$$G(t, u) \approx \Phi(t, u, \alpha) = \sum_{i=1}^{n} \left(B\theta \left(A \begin{pmatrix} t \\ \pi_n u \end{pmatrix} + a \right) \right)_i e_i$$

where

- (e_i) is an ONB of $L^2(\Lambda)$, $\pi_n u = (\langle u, e_1 \rangle, \dots, \langle u, e_n \rangle)^\top$
- $\alpha = (A, a, B) \in \mathbb{R}^{k \times (n+1)} \times \mathbb{R}^k \times \mathbb{R}^{n \times k}$
- θ activator function.

Reduced Control Problem

Leads to controlled SPDE

$$\begin{cases} \mathrm{d}u_t^\alpha = [\Delta u_t^\alpha + b(u_t^\alpha) + \Phi(t, u_t^\alpha, \alpha)]\mathrm{d}t + \sigma \mathrm{d}W_t \\ u_0^\alpha = u \in L^2(\Lambda), \end{cases}$$

where $\alpha \in \mathbb{R}^d$, and cost functional $J : \mathbb{R}^d \to \mathbb{R}$,

$$J(\alpha) = \mathbb{E}\left[\int_0^T \int_\Lambda l(t, x, u_t^\alpha(x))\mathrm{d}x + \frac{\nu}{2}\|\Phi(t, u_t^\alpha, \alpha)\|_{L^2(\Lambda)}^2 \mathrm{d}t + \int_\Lambda m(x, u_T^\alpha(x))\mathrm{d}x\right].$$

Find gradient:

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left(J(\alpha + \varepsilon\beta) - J(\alpha)\right) = \langle \nabla J(\alpha), \beta \rangle.$$

Adjoint Equation

Linearized state equation

$$\begin{cases} \mathrm{d}v_t^\beta = [\Delta v_t^\beta + b_u(u_t^\alpha)v_t^\beta + \Phi_u(t, u^\alpha, \alpha)v_t^\beta + \Phi_\alpha(t, u_t^\alpha, \alpha)\beta]\mathrm{d}t \\ v_0^\beta = 0. \end{cases}$$

Adjoint state equation:

$$\begin{cases} \mathrm{d}p_t = -[(\Delta + b_u(u_t^\alpha) + \Phi_u^*(t, u_t^\alpha, \alpha))p_t + l_u(t, u_t^\alpha) + \nu\Phi_u^*(t, u_t^\alpha, \alpha)\Phi(t, u_t^\alpha, \alpha)]\mathrm{d}t \\ p_T = m_u(u_T^\alpha). \end{cases}$$

Adjoint state property:

$$\begin{aligned} &\mathbb{E}\left[\int_0^T \langle \Phi_\alpha(t, u_t^\alpha, \alpha)\beta, p_t\rangle \mathrm{d}t\right] \\ &= \mathbb{E}\left[\int_0^T \langle l_u(t, u_t^\alpha), v_t^\beta\rangle + \nu\langle \Phi(t, u_t^\alpha, \alpha), \Phi_u(t, u_t^\alpha, \alpha)v_t^\beta\rangle \mathrm{d}t + \langle m_u(u_T^\alpha), v_T^\beta\rangle\right] \end{aligned}$$

Gradient of Cost Functional

Directional derivative of cost functional:

$$
\begin{aligned}
\frac{\partial J(\alpha)}{\partial \beta} =& \mathbb{E}\Big[\int_0^T \langle l_u(t, u_t^\alpha), v_t^\beta \rangle + \nu \langle \Phi(t, u_t^\alpha, \alpha), \Phi_\alpha(t, u_t^\alpha, \alpha)\beta \rangle \\
& \qquad + \nu \langle \Phi(t, u_t^\alpha, \alpha), \Phi_u(t, u_t^\alpha, \alpha) v_t^\beta \rangle \mathrm{d}t + \langle m_u(u_T^\alpha), v_T^\beta \rangle \Big] \\
=& \langle \nabla J(\alpha), \beta \rangle,
\end{aligned}
$$

where

$$
\nabla J(\alpha) = \mathbb{E}\left[\int_0^T \nu \Phi_\alpha^*(t, u_t^\alpha, \alpha) \Phi(t, u_t^\alpha, \alpha) + \Phi_\alpha^*(t, u_t^\alpha, \alpha) p_t \mathrm{d}t \right].
$$

Gradient Descent Algorithm

1) For control α_n, solve state equation

$$\begin{cases} \mathrm{d}u_t^n = [\Delta u_t^n + b(u_t^n) + \Phi(t, u_t^n, \alpha_n)]\mathrm{d}t + \sigma \mathrm{d}W_t \\ u_0^n = u \in L^2(\Lambda) \end{cases}$$

2) Solve adjoint equation

$$\begin{cases} \mathrm{d}p_t^n = -[(\Delta + b_u(u_t^n) + \Phi_u^*(t, u_t^n, \alpha_n))p_t^n + l_u(t, u_t^n) + \nu \Phi_u^*(t, u_t^n, \alpha_n)\Phi(t, u_t^n, \alpha_n)]\mathrm{d}t \\ p_T = m_u(u_T^n) \end{cases}$$

3) Approximate gradient

$$\nabla J(\alpha_n) = \mathbb{E}\left[\int_0^T \nu \Phi_\alpha^*(t, u_t^n, \alpha_n)\Phi(t, u_t^n, \alpha_n) + \Phi_\alpha^*(t, u_t^n, \alpha_n)p_t^n \mathrm{d}t\right]$$

4) For step size s, compute new control

$$\alpha_{n+1} = \alpha_n - s\nabla J(\alpha_n)$$

Simulations

Consider

$$\begin{cases} \mathrm{d}u_t^\alpha = \left[\Delta u_t^\alpha + u_t^\alpha(u_t^\alpha - 1)\left(\frac{1}{2} - u_t^\alpha\right) + \Phi(t, u_t^\alpha, \alpha)\right]\mathrm{d}t + \sigma \mathrm{d}W_t, \quad t \in [0,100] \\ u_0^\alpha = u \in L^2(0,20) \end{cases}$$

with Neumann boundary conditions and $u = 1_{[5,15]}$.

$$J(\alpha)$$
$$= \mathbb{E}\left[\int_0^{100} \|u_t^\alpha - u_t^0\|_{L^2(0,20)}^2 + \frac{1}{2}\|\Phi(t, u_t^\alpha, \alpha)\|_{L^2(0,20)}^2 \mathrm{d}t + \|u_{100}^\alpha - u_{100}^0\|_{L^2(0,20)}^2\right],$$

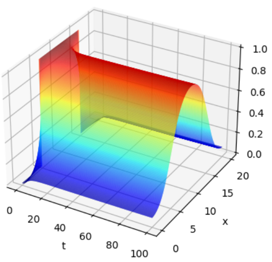
where u^0 is the solution without control and noise.

Simulations

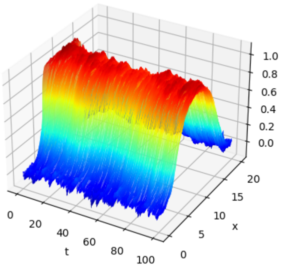
Consider

$$\begin{cases} \mathrm{d}u_t^\alpha = \left[\Delta u_t^\alpha + u_t^\alpha(u_t^\alpha - 1)\left(\frac{1}{2} - u_t^\alpha\right) + \Phi(t, u_t^\alpha, \alpha)\right]\mathrm{d}t + \sigma \mathrm{d}W_t, \quad t \in [0, 100] \\ u_0^\alpha = u \in L^2(0, 20) \end{cases}$$

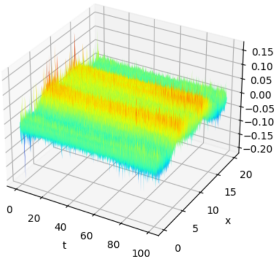
with Neumann boundary conditions and $u = 1_{[5,15]}$.



reference profile
u^0 $(\sigma = 0)$



controlled solution
$\sigma = 0.05$

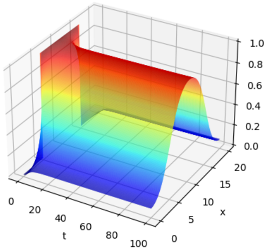


approximated control
$\sigma = 0.05$

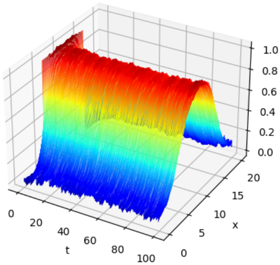
Radial Basis Function Network

Consider feedback control of the type

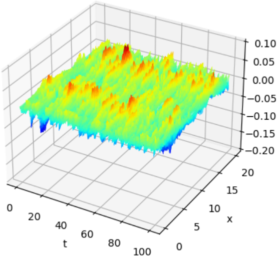
$$G(t,u)(x) \approx \Phi(t,u,\alpha,\tilde{u})(x) = \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{r} \alpha_{ijk} 1_{[t_{k-1},t_k]}(t) e^{-\kappa|u(x)-\tilde{u}_j|^2} e_i(x).$$



reference profile
u^0 $(\sigma = 0)$



controlled solution
$\sigma = 0.05$



approximated control
$\sigma = 0.05$

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