Controlled SPDEs: Peng's Maximum Principle and Numerical Methods

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Stochastic Control & Financial Engineering, Princeton University, June 22, 2023

Deutscher Akademischer Austauschdienst
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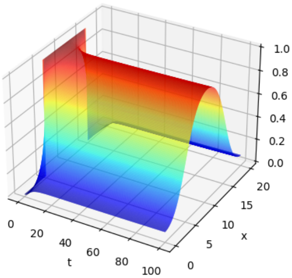
Example

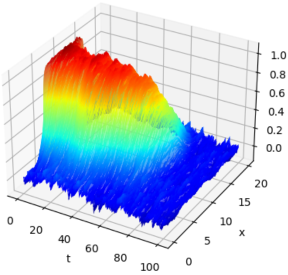
Consider the Nagumo equation

$$\begin{cases} \mathrm{d}u_t = \left[\Delta u_t + u_t(u_t - 1)\left(\frac{1}{2} - u_t\right)\right]\mathrm{d}t + \sigma \mathrm{d}W_t, & t \in [0, 100] \\ u_0 = u \in L^2(0, 20) \end{cases}$$

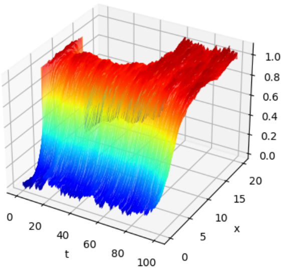
with Neumann boundary conditions and $u = 1_{[5,15]}$.



$\sigma = 0$



$\sigma = 0.05$



$\sigma = 0.05$

Control of the Stochastic Nagumo Equation

Introduce control $g : [0,100] \times [0,20] \times \Omega \to \mathbb{R}$

$$\begin{cases} \mathrm{d}u_t^g = \left[\Delta u_t^g + u_t^g(u_t^g - 1)\left(\frac{1}{2} - u_t^g\right) + g_t\right]\mathrm{d}t + \sigma \mathrm{d}W_t, \quad t \in [0,100] \\ u_0^g = u \in L^2(0,20) \end{cases} \tag{$\star$}$$

and cost functional

$$J(g) = \mathbb{E}\left[\int_0^{100}\int_\Lambda \left(u_t^g(x) - u^{\mathrm{ref}}(t,x)\right)^2 + g^2(t,x)\mathrm{d}x\mathrm{d}t + \int_\Lambda \left(u_{100}^g(x) - u^T(x)\right)^2 \mathrm{d}x\right]$$

where u^{ref}, u^T are desired reference profiles.

Goal:

$$\boxed{\text{Minimize} \quad J \quad \text{subject to } (\star)}$$

Outline

Setting

Minimize

$$J(g) := \mathbb{E}\left[\int_0^T \int_\Lambda l(u_t^g(x), g_t)\mathrm{d}x\mathrm{d}t + \int_\Lambda m(u_T^g(x))\mathrm{d}x\right]$$

subject to

$$\begin{cases} \mathrm{d}u_t^g = [\Delta u_t^g + b(u_t^g, g_t)]\mathrm{d}t + \sigma(u_t^g, g_t)\mathrm{d}W_t, \quad t \in [0, T] \\ u_0^g = u_0 \in L^2(\Lambda), \end{cases}$$

where

- l, m Nemytskii operators of (at most) quadratic growth
- b, σ Nemytskii operators, Lipschitz
- $(W_t)_{t\geq 0}$ cylindrical Wiener process
- $\Lambda \subset \mathbb{R}^d$ bounded domain
- control domain $\mathcal{U}$ non-convex.

Setting

Minimize

$$J(g) := \mathbb{E}\left[\int_0^T \int_\Lambda l(u_t^g(x), g_t)\mathrm{d}x\mathrm{d}t + \int_\Lambda m(u_T^g(x))\mathrm{d}x\right]$$

subject to

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- Pontryagin's maximum principle (1956): Necessary optimality condition for controlled ODEs.
- Peng's maximum principle (1990): Generalization to controlled SDEs.
- Since then: Generalizations to controlled SPDEs by Du, Fuhrman, Frankowska, Guatteri, Hu, Li, Lü, Meng, Tang, Tessitore, Zhang, ...
  Previous results require strong assumptions on coefficients l and h excluding quadratic costs.

Spike Variation

Let $(\bar{u}, \bar{g})$ be optimal. Fix $\tau \in [0, T)$, $\varepsilon > 0$, $h \in U$, and set

$$g_t^\varepsilon := \begin{cases} h, & \tau \leq t \leq \tau + \varepsilon, \\ \bar{g}_t & \text{otherwise.} \end{cases}$$

Then

$$0 \leq J(g^\varepsilon) - J(\bar{g}) = \mathbb{E}\left[\int_0^T \int_\Lambda l(u_t^\varepsilon, g_t^\varepsilon) - l(\bar{u}_t, \bar{g}_t) \mathrm{d}x \mathrm{d}t + \int_\Lambda m(u_T^\varepsilon) - m(\bar{u}_T) \mathrm{d}x \right].$$

Roadmap: 1) Taylor expand integrands;
2) Divide by ε, send $\varepsilon \to 0$;
3) Identify remaining terms.

Because of stochastic calculus, we have to Taylor expand up to second order.

Quadratic Terms

Taylor expanding the cost functional to second order leads to the quadratic terms

$$\mathbb{E}\left[\int_0^T \int_\Lambda l_{uu}(\bar{u}_t(x), \bar{g}_t) v_t^\varepsilon(x) v_t^\varepsilon(x) \mathrm{d}x\mathrm{d}t + \int_\Lambda m_{uu}(\bar{u}_T(x)) v_T^\varepsilon(x) v_T^\varepsilon(x) \mathrm{d}x\right].$$

Peng's idea: Linearize using tensor product and derive equation for $v_t^\varepsilon \otimes v_t^\varepsilon$ on

$$\mathbb{R}^n \otimes \mathbb{R}^n \cong \mathbb{R}^{n\times n}.$$

Infinite dimensional analogue:

$$H \otimes H \cong L_2(H).$$

Problem: Quadratic costs require duality analysis on $L(H) \cong L_1(H)^*$.

Explicit Tensor Product

Instead, we use explicit representation

$$
\begin{aligned}
L^2(\Lambda) \otimes L^2(\Lambda) &\cong L^2(\Lambda^2) \\
v \otimes w &\leftrightsquigarrow ((x, y) \mapsto v(x)w(y)) .
\end{aligned}
$$

Rewrite quadratic terms as

$$
\begin{aligned}
&\mathbb{E}\left[\int_0^T \int_\Lambda l_{uu}(\bar{u}_t(x), \bar{g}_t) v_t^\varepsilon(x) v_t^\varepsilon(x) \mathrm{d}x \mathrm{d}t\right] \\
&= \mathbb{E}\left[\int_0^T \int_\Lambda l_{uu}(\bar{u}_t(x), \bar{g}_t) \delta(v_t^\varepsilon \otimes v_t^\varepsilon)(x) \mathrm{d}x \mathrm{d}t\right],
\end{aligned}
$$

where

$$
\begin{aligned}
\delta : H_0^1(\Lambda^2) &\to L^2(\Lambda) \\
V &\mapsto (x \mapsto V(x, x)).
\end{aligned}
$$

Second Order Adjoint Equation

Theorem (Stannat, W., SICON 2021)

The equation

$$\begin{cases} \mathrm{d}P_t(x,y) = -[\Delta P_t(x,y) + (b_u(\bar{u}_t(x),\bar{g}_t) + b_u(\bar{u}_t(y),\bar{g}_t))P_t(x,y) \\ \qquad\qquad\qquad + \langle \sigma_u(\bar{u}_t(x),\bar{g}_t), \sigma_u(\bar{u}_t(y),\bar{g}_t)\rangle_{L_2(\Xi,\mathbb{R})} P_t(x,y) \\ \qquad\qquad\qquad + \langle \sigma_u(\bar{u}_t(x),\bar{g}_t) + \sigma_u(\bar{u}_t(y),\bar{g}_t), Q_t(x,y)\rangle_{L_2(\Xi,\mathbb{R})} \\ \qquad\qquad\qquad + \delta^*(l_{uu}(\bar{u}_t(x),\bar{g}_t)) + \delta^*(b_{uu}(\bar{u}_t(x),\bar{g}_t)p_t(x)) \\ \qquad\qquad\qquad + \delta^*(\langle \sigma_{uu}(\bar{u}_t(x),\bar{g}_t), q_t\rangle_{L_2(\Xi,\mathbb{R})})]\mathrm{d}t + Q_t(x,y)\mathrm{d}W_t \\ P_T(x,y) = \delta^*(m_{uu}(\bar{u}_T(x))) \end{cases}$$

has a unique adapted solution (P,Q), *where*

$$P \in L^2([0,T]\times\Omega; L^2(\Lambda^2)) \cap L^2(\Omega; C([0,T]; H^{-1}(\Lambda^2))),$$

and

$$Q \in L^2([0,T]\times\Omega; L_2(\Xi; H^{-1}(\Lambda^2))).$$

Peng's Maximum Principle for SPDEs

Theorem (Stannat, W., SICON 2021)

Let $(\bar{u}, \bar{g})$ *be an optimal pair. Then there exist adapted processes*

$$(p, q) \in L^2([0,T] \times \Omega; H_0^1(\Lambda)) \times L^2([0,T] \times \Omega; L_2(\Xi, L^2(\Lambda)))$$

satisfying the first order adjoint equation and adapted processes

$$(P, Q) \in L^2([0,T] \times \Omega; L^2(\Lambda^2)) \times L^2([0,T] \times \Omega; L_2(\Xi, H^{-1}(\Lambda^2)))$$

satisfying the second order adjoint equation such that

$$\inf_{g \in \mathcal{U}} \mathcal{G}(t, \bar{u}_t, g) = \mathcal{G}(t, \bar{u}_t, \bar{g}_t),$$

for almost all $(t, \omega) \in [0,T] \times \Omega$*, where* $\mathcal{G} : [0,T] \times L^2(\Lambda) \times \mathcal{U} \to \mathbb{R}$

$$\begin{aligned} \mathcal{G}(t,u,g) := \int_\Lambda l(u(x), g)\mathrm{d}x + \langle p_t, b(u,g)\rangle_{L^2(\Lambda)} + tr\left(\sigma(u,g)^* q_t\right) \\ + \frac{1}{2} tr\left(\sigma(u,g)^* P_t \sigma(u,g)\right) - tr\left(\sigma(u,g)^* P_t \sigma(\bar{u}_t, \bar{g}_t)\right). \end{aligned}$$

Feedback Controls

Consider feedback control

$$G : [0, T] \times L^2(\Lambda) \to L^2(\Lambda)$$

and controlled SPDE

$$\begin{cases} \mathrm{d}u_t^G = [\Delta u_t^G + b(u_t^G) + G(t, u_t^G)]\mathrm{d}t + \sigma \mathrm{d}W_t, \ t \in [0, T] \\ u_0^G = u \in L^2(\Lambda). \end{cases}$$

Use finite dimensional approximation (e.g. neural network)

$$G(t, u) \approx \Phi(t, u, \alpha) = \sum_{i=1}^{n} \left(B\theta \left(A \begin{pmatrix} t \\ \pi_n u \end{pmatrix} + a \right) \right)_i e_i$$

where

- (e_i) is an ONB of $L^2(\Lambda)$, $\pi_n u = (\langle u, e_1 \rangle, \dots, \langle u, e_n \rangle)^\top$
- $\alpha = (A, a, B) \in \mathbb{R}^{k \times (n+1)} \times \mathbb{R}^k \times \mathbb{R}^{n \times k}$
- θ activator function.

Reduced Control Problem

Leads to controlled SPDE

$$\begin{cases} \mathrm{d}u_t^\alpha = [\Delta u_t^\alpha + b(u_t^\alpha) + \Phi(t, u_t^\alpha, \alpha)]\mathrm{d}t + \sigma \mathrm{d}W_t \\ u_0^\alpha = u \in L^2(\Lambda), \end{cases}$$

where $\alpha \in \mathbb{R}^d$, and cost functional $J : \mathbb{R}^d \to \mathbb{R}$,

$$J(\alpha) = \mathbb{E}\left[\int_0^T \int_\Lambda l(t, x, u_t^\alpha(x))\mathrm{d}x + \frac{\nu}{2}\|\Phi(t, u_t^\alpha, \alpha)\|_{L^2(\Lambda)}^2 \mathrm{d}t + \int_\Lambda m(x, u_T^\alpha(x))\mathrm{d}x\right].$$

Find gradient:

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left(J(\alpha + \varepsilon\beta) - J(\alpha)\right) = \langle \nabla J(\alpha), \beta \rangle.$$

Adjoint Equation

Linearized state equation

$$\begin{cases} \mathrm{d}v_t^\beta = [\Delta v_t^\beta + b_u(u_t^\alpha)v_t^\beta + \Phi_u(t,u^\alpha,\alpha)v_t^\beta + \Phi_\alpha(t,u_t^\alpha,\alpha)\beta]\mathrm{d}t \\ v_0^\beta = 0. \end{cases}$$

Adjoint state equation:

$$\begin{cases} \mathrm{d}p_t = -[(\Delta + b_u(u_t^\alpha) + \Phi_u^*(t,u_t^\alpha,\alpha))p_t + l_u(t,u_t^\alpha) + \nu\Phi_u^*(t,u_t^\alpha,\alpha)\Phi(t,u_t^\alpha,\alpha)]\mathrm{d}t \\ p_T = m_u(u_T^\alpha). \end{cases}$$

Adjoint state property:

$$\begin{aligned} &\mathbb{E}\left[\int_0^T \langle \Phi_\alpha(t,u_t^\alpha,\alpha)\beta, p_t\rangle \mathrm{d}t\right] \\ &= \mathbb{E}\left[\int_0^T \langle l_u(t,u_t^\alpha), v_t^\beta\rangle + \nu\langle \Phi(t,u_t^\alpha,\alpha), \Phi_u(t,u_t^\alpha,\alpha)v_t^\beta\rangle \mathrm{d}t + \langle m_u(u_T^\alpha), v_T^\beta\rangle\right] \end{aligned}$$

Gradient of Cost Functional

Directional derivative of cost functional:

$$\begin{aligned}\frac{\partial J(\alpha)}{\partial \beta} =& \mathbb{E}\Big[\int_0^T \langle l_u(t,u_t^\alpha), v_t^\beta\rangle + \nu\langle \Phi(t,u_t^\alpha,\alpha), \Phi_\alpha(t,u_t^\alpha,\alpha)\beta\rangle \\ &\qquad + \nu\langle \Phi(t,u_t^\alpha,\alpha), \Phi_u(t,u_t^\alpha,\alpha)v_t^\beta\rangle \mathrm{d}t + \langle m_u(u_T^\alpha), v_T^\beta\rangle\Big] \\ =& \langle \nabla J(\alpha), \beta\rangle.\end{aligned}$$

Theorem (Stannat, Vogler, W. (2023+))

The gradient is given by

$$\nabla J(\alpha) = \mathbb{E}\left[\int_0^T \nu\Phi_\alpha^*(t,u_t^\alpha,\alpha)\Phi(t,u_t^\alpha,\alpha) + \Phi_\alpha^*(t,u_t^\alpha,\alpha)p_t \mathrm{d}t\right],$$

where p is the adjoint state.

Gradient Descent Algorithm

1) For control α_n, solve state equation

$$\begin{cases} \mathrm{d}u_t^n = [\Delta u_t^n + b(u_t^n) + \Phi(t, u_t^n, \alpha_n)]\mathrm{d}t + \sigma \mathrm{d}W_t \\ u_0^n = u \in L^2(\Lambda) \end{cases}$$

2) Solve adjoint equation

$$\begin{cases} \mathrm{d}p_t^n = -[(\Delta + b_u(u_t^n) + \Phi_u^*(t, u_t^n, \alpha_n))p_t^n + l_u(t, u_t^n) + \nu \Phi_u^*(t, u_t^n, \alpha_n)\Phi(t, u_t^n, \alpha_n)]\mathrm{d}t \\ p_T = m_u(u_T^n) \end{cases}$$

3) Approximate gradient

$$\nabla J(\alpha_n) = \mathbb{E}\left[\int_0^T \nu \Phi_\alpha^*(t, u_t^n, \alpha_n)\Phi(t, u_t^n, \alpha_n) + \Phi_\alpha^*(t, u_t^n, \alpha_n)p_t^n \mathrm{d}t\right]$$

4) For step size s, compute new control

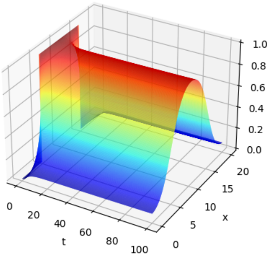
$$\alpha_{n+1} = \alpha_n - s\nabla J(\alpha_n)$$

Simulations

Consider

$$\begin{cases} \mathrm{d}u_t^\alpha = \left[\Delta u_t^\alpha + u_t^\alpha(u_t^\alpha - 1)\left(\frac{1}{2} - u_t^\alpha\right) + \Phi(t, u_t^\alpha, \alpha)\right] \mathrm{d}t + \sigma \mathrm{d}W_t, \quad t \in [0, 100] \\ u_0^\alpha = u \in L^2(0, 20) \end{cases}$$

with Neumann boundary conditions and $u = 1_{[5,15]}$.



uncontrolled solution u^0
($\sigma = 0$)

Simulations

Consider

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with Neumann boundary conditions and $u = 1_{[5,15]}$.

$$J(\alpha)$$
$$= \mathbb{E}\left[\int_0^{100} \|u_t^\alpha - u_t^0\|^2_{L^2(0,20)} + \frac{1}{2}\|\Phi(t, u_t^\alpha, \alpha)\|^2_{L^2(0,20)}\mathrm{d}t + \|u_{100}^\alpha - u_{100}^0\|^2_{L^2(0,20)}\right].$$

Simulations

Consider

$$\begin{cases} \mathrm{d}u_t^\alpha = \left[\Delta u_t^\alpha + u_t^\alpha(u_t^\alpha - 1)\left(\frac{1}{2} - u_t^\alpha\right) + \Phi(t, u_t^\alpha, \alpha)\right]\mathrm{d}t + \sigma \mathrm{d}W_t, & t \in [0,100] \\ u_0^\alpha = u \in L^2(0,20) \end{cases}$$

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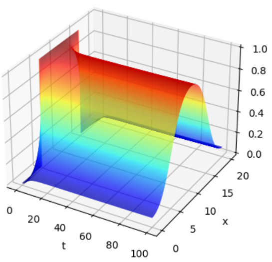
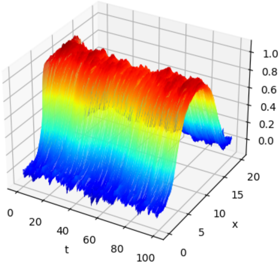
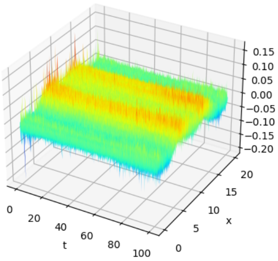


reference profile
u^0 $(\sigma = 0)$



controlled solution
$\sigma = 0.05$

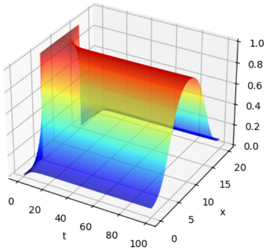


approximated control
$\sigma = 0.05$

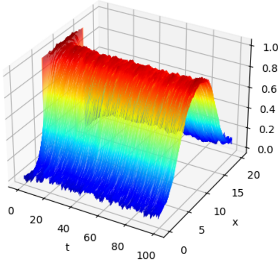
Radial Basis Function Network

Consider feedback control of the type

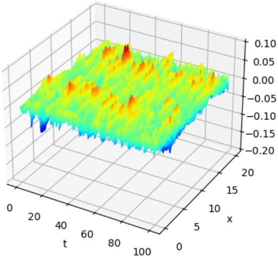
$$G(t,u)(x) \approx \Phi(t,u,\alpha,\tilde{u})(x) = \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{r} \alpha_{ijk} 1_{[t_{k-1},t_k]}(t) e^{-\kappa|u(x)-\tilde{u}_j|^2} e_i(x).$$



reference profile
u^0 $(\sigma = 0)$



controlled solution
$\sigma = 0.05$



approximated control
$\sigma = 0.05$

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