Stochastic optimal control in Hilbert spaces: $C^{1,1}$ regularity of the value function and optimal synthesis

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Introduction

Minimize

$$J(t,x;a(\cdot)) := \mathbb{E}\left[\int_t^T l(X(s),a(s))\mathrm{d}s + g(X(T))\right]$$

over admissible controls $a(\cdot) : [t,T] \times \Omega \to \Lambda_0$ subject to

$$\begin{cases} \mathrm{d}X(s) = [AX(s) + b(X(s),a(s))]\mathrm{d}s + \sigma(X(s),a(s))\mathrm{d}W(s), & s \in [t,T] \\ X(t) = x \in H, \end{cases}$$

where

- $l : H \times \Lambda_0 \to \mathbb{R}$ are running cost
- $g : H \to \mathbb{R}$ are terminal cost
- $A : \mathcal{D}(A) \subset H \to H$ linear unbounded operator
- b and σ drift and diffusion coefficient
- $(W(s))_{s\in[t,T]}$ cylindrical Wiener process

Example: Stochastic Partial Differential Equations

Consider

$$\begin{cases} A := \sum_{i,j=1}^{d} \partial_i(a_{ij}\partial_j) + \sum_{i=1}^{d} b_i\partial_i + c \\ \mathcal{D}(A) = H_0^1(\mathcal{O}) \cap H^2(\mathcal{O}) \end{cases}$$

where $a_{ij} = a_{ji}, b_i \in W^{1,\infty}(\mathcal{O})$, $i,j = 1,\dots,d$, $c \in L^\infty(\mathcal{O})$, and for some $\theta > 0$

$$\sum_{i,j=1}^{d} a_{ij}\xi_i\xi_j \geq \theta|\xi|^2.$$

Then A satisfies:

Assumption (Strong B-condition)

There is a strictly positive, self-adjoint operator $B \in L(H)$ such that $A^*B \in L(H)$ and for some $c_0 \geq 0$

$$-A^*B + c_0 B \geq I.$$

Example: Stochastic Delay Differential Equations

Consider

$$\begin{cases} \mathrm{d}y(s) = b_0\left(y(s), \int_{-d}^{0} \eta_1(\xi) y(s+\xi)\mathrm{d}\xi, a(s)\right)\mathrm{d}s \\ \qquad\qquad +\sigma_0\left(y(s), \int_{-d}^{0} \eta_2(\xi) y(s+\xi)\mathrm{d}\xi, a(s)\right)\mathrm{d}W(s), \\ y(t) = x_0, \quad y(t+\xi) = x_1(\xi) \quad \forall \xi \in [-d,0). \end{cases}$$

Let $H := \mathbb{R}^n \times L^2([-d,0];\mathbb{R}^n)$. Define

$$Ax := \begin{bmatrix} -x_0 \\ x_1' \end{bmatrix}, \quad D(A) := \left\{ x = (x_0, x_1) \in H : x_1 \in W^{1,2}([-d,0];\mathbb{R}^n),\ x_1(0) = x_0 \right\},$$

$$b(x,a) := \begin{bmatrix} b_0\left(x_0, \int_{-d}^{0} \eta_1(\xi) x_1(\xi)\mathrm{d}\xi, a\right) + x_0 \\ 0 \end{bmatrix},$$

$$\sigma(x,u)w := \begin{bmatrix} \sigma_0\left(x_0, \int_{-d}^{0} \eta_2(\xi) x_1(\xi)\mathrm{d}\xi, a\right) w \\ 0 \end{bmatrix}.$$

Remark

Weak B-condition is satisfied for A.

Dynamic Programming

Minimize

$$J(t,x;a(\cdot)) := \mathbb{E}\left[\int_t^T l(X(s),a(s))\mathrm{d}s + g(X(T))\right]$$

over admissible controls $a(\cdot) : [t, T] \times \Omega \to \Lambda_0$ subject to

$$\begin{cases} \mathrm{d}X(s) = [AX(s) + b(X(s),a(s))]\mathrm{d}s + \sigma(X(s),a(s))\mathrm{d}W(s), \quad s \in [t, T] \\ X(t) = x \in H, \end{cases}$$

Define value function

$$V(t,x) := \inf_{a(\cdot)} J(t,x;a(\cdot)).$$

Satisfies dynamic programming principle

$$V(t,x) = \inf_{a(\cdot)} \mathbb{E}\left[\int_t^\tau l(X(s),a(s))\mathrm{d}s + V(\tau, X(\tau))\right], \qquad \forall \tau \in [t, T].$$

Next: Derive Hamilton–Jacobi–Bellman equation.

Derivation of the HJB Equation

Assume V is sufficiently regular. Then

$$\begin{aligned}
V(\tau, X(\tau)) &= V(t,x) + \int_t^\tau V_t(s, X(s))\mathrm{d}s + \int_t^\tau \langle DV(s,X(s)), \mathrm{d}X(s)\rangle \\
&\quad + \frac{1}{2}\int_t^\tau D^2V(s,X(s))\mathrm{d}\langle X\rangle_s \\
&= V(t,x) + \int_t^\tau V_t(s,X(s)) + \langle DV(s,X(s)), AX(s)\rangle \mathrm{d}s \\
&\quad + \int_t^\tau \langle DV(s,X(s)), b(X(s),a(s))\rangle \mathrm{d}s \\
&\quad + \int_t^\tau \frac{1}{2}\mathrm{Tr}[\sigma(X(s),a(s))\sigma^*(X(s),a(s))D^2V(s,X(s))]\mathrm{d}s \\
&\quad + \int_t^\tau DV(s,X(s))\sigma(X(s),a(s))\mathrm{d}W(s).
\end{aligned}$$

Derivation of the HJB Equation

Plugging this into the dynamic programming principle, we obtain

$$\begin{aligned}0 &= \inf_{a(\cdot)} \mathbb{E}\left[\int_t^\tau l(X(s), a(s))\mathrm{d}s + V(\tau, X(\tau)) - V(t,x)\right]\\ &= \inf_{a(\cdot)} \mathbb{E}\bigg[\int_t^\tau l(X(s), a(s)) + V_t(s, X(s)) + \langle DV(s, X(s)), b(X(s), a(s))\rangle\\ &\qquad\qquad + \mathrm{Tr}[\sigma(X(s), a(s))\sigma^*(X(s), a(s))D^2V(s, X(s))]\mathrm{d}s\bigg].\end{aligned}$$

Dividing by $\tau - t$ and sending $\tau \downarrow t$ yields

$$\begin{cases} V_t(t,x) + \langle Ax, DV(t,x)\rangle \\ \quad + \inf_a \left\{ l(x,a) + \langle DV(t,x), b(x,a)\rangle + \frac{1}{2}\mathrm{Tr}[\sigma(x,a)\sigma^*(x,a)D^2V(t,x)]\right\} = 0 \\ V(T,x) = g(x). \end{cases}$$

Verification Theorem

Theorem

Assume V is sufficiently regular. If

$$a^*(s) \in \arg\min_a \Big\{ l(X^*(s), a) + \langle DV(s, X^*(s)), b(X^*(s), a) \rangle + \frac{1}{2} Tr[\sigma(X^*(s), a)\sigma^*(X^*(s), a) D^2 V(t, x)] \Big\},$$

then a^* is optimal.

Remark

Assume $l(x, a) = l_1(x) + l_2(a)$. Then, under certain assumptions,

$$a^*(s) = Dl_2^{-1}(DV(s, X^*(s))).$$

is optimal.

Viscosity Solutions – Motivation

Consider one-dimensional HJB equation

$$\begin{cases} V_t(t,x) + \inf_a \left\{ l(x,a) + V_x(t,x)b(x,a) + \frac{1}{2}V_{xx}(t,x)\sigma^2(x,a) \right\} = 0 \\ V(T,x) = g(x), \quad x \in \mathbb{R}. \end{cases}$$

Let $\varphi \in C^\infty((0,T)\times\mathbb{R})$ be such that $V-\varphi$ has global maximum at (t,x), i.e.,

$$V(s,y) - V(t,x) \leq \varphi(s,y) - \varphi(t,x)$$

for all $(s,y) \in [0,T]\times\mathbb{R}$. Then,

$$\begin{aligned} 0 &= \inf_a \mathbb{E}\left[\int_t^\tau l(X_s,a_s)\mathrm{d}s + V(\tau,X_\tau) - V(t,x)\right] \\ &\leq \inf_a \mathbb{E}\left[\int_t^\tau l(X_s,a_s)\mathrm{d}s + \varphi(\tau,X_\tau) - \varphi(t,x)\right]. \end{aligned}$$

Thus,

$$\varphi_t(t,x) + \inf_a \left\{ l(x,a) + \varphi_x(t,x)b(x,a) + \frac{1}{2}\varphi_{xx}(t,x)\sigma^2(x,a) \right\} \geq 0.$$

Viscosity Solutions – Definition

Consider the HJB equation

$$\begin{cases} V_t(t,x) + \inf_a \left\{ l(x,a) + V_x(t,x)b(x,a) + \frac{1}{2} V_{xx}(t,x)\sigma^2(x,a) \right\} = 0 \\ V(T,x) = g(x). \end{cases} \tag{1}$$

Definition (Viscosity Solution)

$V \in C([0,T] \times \mathbb{R})$ *is a viscosity subsolution of* (1), *if* $V(T,y) \leq g(y)$ *and* $\forall \varphi \in C^\infty((0,T) \times \mathbb{R})$ *such that* $V - \varphi$ *has a global maximum at* (t,x), *it holds*

$$\varphi_t(t,x) + \inf_a \left\{ l(x,a) + \varphi_x(t,x)b(x,a) + \frac{1}{2}\varphi_{xx}(t,x)\sigma^2(x,a) \right\} \geq 0$$

Viscosity Solutions – Consistency

Let $V \in C^{1,2}((0,T)\times\mathbb{R})$ be a classical solution of

$$\begin{cases} V_t(t,x) + \inf_a \left\{ l(x,a) + V_x(t,x)b(x,a) + \frac{1}{2}V_{xx}(t,x)\sigma^2(x,a)\right\} = 0 \\ V(T,x) = g(x) \end{cases}$$

and let $\varphi \in C^\infty((0,T)\times\mathbb{R})$ be such that $V-\varphi$ has a global maximum at (t,x). Then

$$V_t(t,x) = \varphi_t(t,x), \quad V_x(t,x) = \varphi_x(t,x), \quad V_{xx}(t,x) \le \varphi_{xx}(t,x).$$

Hence,

$$\begin{aligned} &\varphi_t(t,x) + \inf_a \left\{ l(x,a) + \varphi_x(t,x)b(x,a) + \frac{1}{2}\varphi_{xx}(t,x)\sigma^2(x,a)\right\} \\ &\ge V_t(t,x) + \inf_a \left\{ l(x,a) + V_x(t,x)b(x,a) + \frac{1}{2}V_{xx}(t,x)\sigma^2(x,a)\right\} = 0 \end{aligned}$$

$\implies$ Every classical solution is a viscosity solution.

Viscosity Solutions in Infinite Dimensions

HJB equation:

$$\begin{cases} V_t(t,x) + \langle Ax, DV(t,x)\rangle \\ \quad + \inf_a \left\{ l(x,a) + \langle DV(t,x), b(x,a)\rangle + \frac{1}{2}\mathrm{Tr}[\sigma(x,a)\sigma^*(x,a)D^2V(t,x)] \right\} = 0 \\ V(T,x) = g(x). \end{cases}$$

Follow same idea as in finite-dimensional case: Take test function φ and replace derivatives of V by derivatives of φ.

Problematic term:

$$\langle Ax, D\varphi(t,x)\rangle.$$

$\rightsquigarrow$ restrict class of test functions.

Optimal Synthesis

Next steps:

1. Prove higher regularity of the value function.
2. Construct optimal feedbacks.

For step 1:

Theorem (Lasry, Lions 1986)

Let $v : H \to \mathbb{R}$ be semiconvex and semiconcave. Then $v \in C^{1,1}(H)$.

Definition

$v : H \to \mathbb{R}$ is semiconcave if

$$\lambda v(x) + (1-\lambda)v(x') - v(\lambda x + (1-\lambda)x') \leq C\lambda(1-\lambda)\|x - x'\|_H^2$$

for all $\lambda \in [0,1]$ and $x, x' \in H$.

Remark

v semiconcave if and only if $x \mapsto v(x) - C\|x\|^2$ concave for some C.

Semiconcavity of Value Function

Theorem (de Feo, Święch, W. (2023+))

Let

- b, σ, l, g *be Lipschitz in* x *and linearly growing in* (x, a),
- b, σ *be* $C^{1,1}$ *in* x,
- l, g *be semiconcave in* x.

Then, for every $t \in [0, T]$, *the function* $V(t, \cdot)$ *is semiconcave.*

Proof (Sketch).

$$
\begin{aligned}
&\lambda J(t, x_1; a(\cdot)) + (1-\lambda) J(t, x_0; a(\cdot)) - J(t, \lambda x_1 + (1-\lambda) x_0; a(\cdot)) \\
&= \mathbb{E}\left[\int_t^T \lambda l(X_1(s), a(s)) + (1-\lambda) l(X_0(s), a(s)) - l(X_\lambda(s), a(s)) \mathrm{d}s\right] \\
&\quad + \mathbb{E}\left[\lambda g(X_1(T)) + (1-\lambda) g(X_0(T)) - g(X_\lambda(T))\right].
\end{aligned}
$$

Use semiconcavity of l, g and estimates on $\|X_\lambda - X^\lambda\|^2$ and $\|X_1 - X_0\|^2$. □

Semiconvexity of the Value Function

Case 1: Uniform Convexity of Running Cost in Control Variable

Theorem (de Feo, Święch, W. (2023+))

Let

- σ *be independent of* a
- *assume mild regularity assumptions on the coefficients*
- g *be semiconvex*
- $H \times \Lambda \ni (x,a) \mapsto l(x,a) + C\|x\|_H^2 - \nu\|a\|_\Lambda^2$ *be convex*

Then there is a constant ν_0 *such that if* $\nu \geq \nu_0$*, then* $V(t,\cdot)$ *is semiconvex.*

Proof (Sketch).

$$\begin{aligned}
&\varepsilon + \lambda V(t,x_1) + (1-\lambda)V(t,x_0) - V(t,x_\lambda) \\
&\geq \lambda J(t,x_1;a_1^\varepsilon(\cdot)) + (1-\lambda)J(t,x_0;a_0^\varepsilon(\cdot)) - J(t,x_\lambda;a_\lambda^\varepsilon(\cdot))
\end{aligned}$$

Now use stochastic arguments similar to the ones used before. □

Semiconvexity of the Value Function

Case 3: Comparison for the State Equation

Theorem (de Feo, Święch, W. (2023+))

Let $H = L^2(\mathcal{O})$ and let

- *σ be independent of (x, a), b be of Nemytskii type and convex*
- *e^{sA} be positivity preserving*
- *l, g convex and nonincreasing in x.*

Then $V(t, \cdot)$ is convex.

Proof (Sketch).

$$V(t, x_\lambda) \leq J(t, x_\lambda; a_\lambda^\varepsilon(\cdot)) \leq \mathbb{E}\left[\int_t^T l(X^\lambda(s), a_\lambda^\varepsilon(s))ds + g(X^\lambda(T))\right]$$

Now, use monotonicity of l, g and $X_\lambda(s) \geq X^\lambda(s)$. Then, use convexity of l, g. $\square$

Optimal Synthesis

Assumption

Let there be a Lipschitz continuous selection function

$$\gamma : H \times H \to \Lambda_0, \quad (x, p) \mapsto \gamma(x, p) \in \arg\min_a \{\langle p, b(x, a)\rangle + l(x, a)\}.$$

Theorem (de Feo, Święch, W. (2023+))

Let σ be independent of the control. Then the pair $(a^*(s), X^*(s))$, where

$$\begin{cases} a^*(s) = \gamma(X^*(s), DV(s, X^*(s))) \\ X^*(s) = X(s, t, x; a^*(\cdot)) \end{cases}$$

is an optimal couple.

Optimal Synthesis – Proof

Proof.

Consider the linear equation

$$v_t + \langle Ax, Dv\rangle_H + \frac{1}{2}\mathrm{Tr}[\sigma(x)\sigma^*(x)D^2 v] + \langle \tilde{b}(t,x), Dv\rangle_H + \tilde{l}(t,x) = 0,$$

where $\tilde{b}(t,x) := b(x, \gamma(x, DV(t,x)))$ and $\tilde{l}(t,x) := l(x, \gamma(x, DV(t,x)))$.
This equation has unique viscosity solution which is given by

$$v(t,x) = \mathbb{E}\left[\int_t^T \tilde{l}(s, X(s))\mathrm{d}s + g(X(T))\right],$$

where $X(s)$ is the solution of

$$\begin{cases} \mathrm{d}X(s) = [AX(s) + \tilde{b}(s, X(s))]\mathrm{d}s + \sigma(X(s))\mathrm{d}W(s) \\ X(t) = x. \end{cases}$$

$\square$

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