Stochastic Optimal Control in Infinite Dimensions: Verification and Optimal Synthesis

Lukas Wessels

based on joint work with F. de Feo, W. Stannat and A. Święch

www.lukaswessels.org

Seminar on Analysis and Stochastic Analysis
Auburn University, February 21, 2024

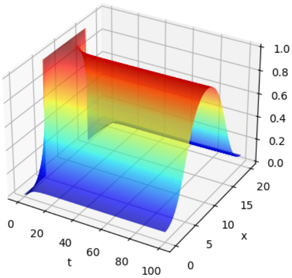
Georgia Institute of Technology

Example

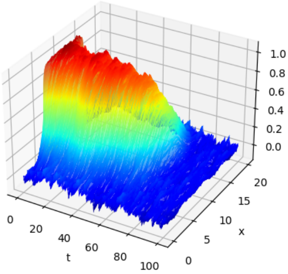
Consider the Nagumo equation

$$\begin{cases} \mathrm{d}X(s) = \left[\Delta X(s) + X(s)(X(s)-1)\left(\frac{1}{2} - X(s)\right)\right]\mathrm{d}s + \sigma \mathrm{d}W_s, & s \in [0,100] \\ X(0) = x \in L^2(0,20) \end{cases}$$

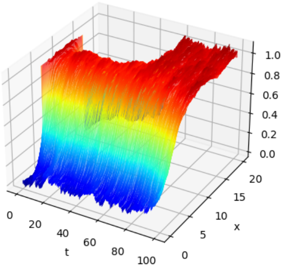
with Neumann boundary conditions and $x = 1_{[5,15]}$.



$\sigma = 0$



$\sigma = 0.05$



$\sigma = 0.05$

Control of the Stochastic Nagumo Equation

Introduce control $a : [0, 100] \times [0, 20] \times \Omega \to \mathbb{R}$

$$\begin{cases} dX(s) = \left[\Delta X(s) + X(s)(X(s) - 1)\left(\frac{1}{2} - X(s)\right) + a(s)\right] ds + \sigma dW_s, & s \in [0, 100] \\ X(0) = x \in L^2(0, 20) \end{cases} \tag{$\star$}$$

and cost functional

$$J(a) = \mathbb{E}\Bigg[\int_0^{100}\int_{\mathcal{O}} \left(X(s,\xi) - X^{\mathrm{ref}}(s,\xi)\right)^2 + a^2(s,\xi)\, d\xi ds + \int_{\mathcal{O}} \left(X(100,\xi) - X^{\mathrm{ref}}(T,\xi)\right)^2 d\xi\Bigg]$$

where X^{ref} is desired reference profile.

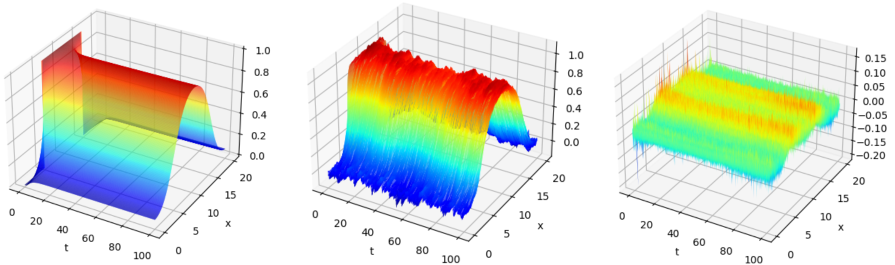
Goal:

Minimize J subject to $(\star)$

Simulations

Introduce control $a : [0, 100] \times [0, 20] \times \Omega \to \mathbb{R}$

$$\begin{cases} dX(s) = \left[\Delta X(s) + X(s)(X(s) - 1)\left(\frac{1}{2} - X(s)\right) + a(s)\right] ds + \sigma dW_s, & s \in [0, 100] \\ X(0) = x \in L^2(0, 20). \end{cases} \quad (\star)$$



reference profile $a = \sigma = 0$ | controlled solution $\sigma = 0.05$ | approximated control $\sigma = 0.05$

General Setting

Minimize

$$J(t, x; a(\cdot)) := \mathbb{E}\left[\int_t^T l(X(s), a(s))\mathrm{d}s + g(X(T))\right]$$

over admissible controls $a(\cdot) : [t, T] \times \Omega \to \Lambda_0$ subject to

$$\begin{cases} \mathrm{d}X(s) = [AX(s) + b(X(s), a(s))]\mathrm{d}s + \sigma(X(s), a(s))\mathrm{d}W(s), & s \in [t, T] \\ X(t) = x \in H, \end{cases}$$

where

- $l : H \times \Lambda_0 \to \mathbb{R}$ are running cost
- $g : H \to \mathbb{R}$ are terminal cost
- $A : \mathcal{D}(A) \subset H \to H$ linear unbounded operator
- b and σ drift and diffusion coefficient
- $(W(s))_{s\in[t,T]}$ cylindrical Wiener process

Dynamic Programming

Minimize

$$J(t, x; a(\cdot)) := \mathbb{E}\left[\int_t^T l(X(s), a(s))\mathrm{d}s + g(X(T))\right]$$

over admissible controls $a(\cdot) : [t, T] \times \Omega \to \Lambda_0$ subject to

$$\begin{cases} \mathrm{d}X(s) = [AX(s) + b(X(s), a(s))]\mathrm{d}s + \sigma(X(s), a(s))\mathrm{d}W(s), & s \in [t, T] \\ X(t) = x \in H, \end{cases}$$

Define **value function**

$$V(t, x) := \inf_{a(\cdot)} J(t, x; a(\cdot)).$$

Satisfies dynamic programming principle

$$V(t, x) = \inf_{a(\cdot)} \mathbb{E}\left[\int_t^\tau l(X(s), a(s))\mathrm{d}s + V(\tau, X(\tau))\right], \qquad \forall \tau \in [t, T].$$

Next: Derive Hamilton–Jacobi–Bellman equation.

Derivation of the HJB Equation

Assume V is sufficiently regular. Then

$$\begin{aligned}
V(\tau, X(\tau)) = V(t,x) &+ \int_t^\tau V_t(s, X(s))\mathrm{d}s + \int_t^\tau \langle DV(s,X(s)), \mathrm{d}X(s)\rangle \\
&+ \frac{1}{2}\int_t^\tau D^2 V(s,X(s))\mathrm{d}\langle X\rangle_s \\
= V(t,x) &+ \int_t^\tau V_t(s,X(s)) + \langle DV(s,X(s)), AX(s)\rangle \mathrm{d}s \\
&+ \int_t^\tau \langle DV(s,X(s)), b(X(s),a(s))\rangle \mathrm{d}s \\
&+ \int_t^\tau \frac{1}{2}\mathrm{Tr}[\sigma(X(s),a(s))\sigma^*(X(s),a(s))D^2V(s,X(s))]\mathrm{d}s \\
&+ \int_t^\tau DV(s,X(s))\sigma(X(s),a(s))\mathrm{d}W(s).
\end{aligned}$$

Plug this into dynamic programming principle.

Derivation of the HJB Equation

Plugging this into the dynamic programming principle, we obtain

$$
\begin{aligned}
0 &= \inf_{a(\cdot)} \mathbb{E}\left[\int_t^\tau l(X(s),a(s))\mathrm{d}s + V(\tau, X(\tau)) - V(t,x)\right] \\
&= \inf_{a(\cdot)} \mathbb{E}\Bigg[\int_t^\tau l(X(s),a(s)) + V_t(s,X(s)) \\
&\qquad\qquad + \langle DV(s,X(s)), b(X(s),a(s))\rangle + \langle DV(s,X(s)), AX(s)\rangle \\
&\qquad\qquad\qquad + \frac{1}{2}\mathrm{Tr}[\sigma(X(s),a(s))\sigma^*(X(s),a(s))D^2V(s,X(s))]\,\mathrm{d}s\Bigg].
\end{aligned}
$$

Dividing by $\tau - t$ and sending $\tau \downarrow t$ yields the HJB equation

$$
\begin{cases}
V_t(t,x) + \langle Ax, DV(t,x)\rangle \\
\quad + \inf_a \left\{ l(x,a) + \langle DV(t,x), b(x,a)\rangle + \frac{1}{2}\mathrm{Tr}[\sigma(x,a)\sigma^*(x,a)D^2V(t,x)] \right\} = 0 \\
V(T,x) = g(x).
\end{cases}
$$

Question: What is all this good for?

Verification and Optimal Synthesis – Smooth Version

Theorem (Verification Theorem)

Assume V is sufficiently regular. If

$$a^*(s) \in \arg\min_a \Big\{ l(X^*(s), a) + \langle DV(s, X^*(s)), b(X^*(s), a) \rangle + \frac{1}{2} Tr[\sigma(X^*(s), a)\sigma^*(X^*(s), a) D^2 V(t, x)] \Big\},$$

then a^* is optimal.

Remark (Optimal Synthesis)

Assume $l(x, a) = l_1(x) + l_2(a)$. Then, under certain assumptions,

$$a^*(s) = Dl_2^{-1}(DV(s, X^*(s))).$$

is optimal.

Viscosity Solutions – Motivation

Consider one-dimensional HJB equation

$$\begin{cases} V_t(t,x) + \inf_a \left\{ l(x,a) + V_x(t,x)b(x,a) + \frac{1}{2}V_{xx}(t,x)\sigma^2(x,a) \right\} = 0 \\ V(T,x) = g(x), \quad x \in \mathbb{R}. \end{cases}$$

Let $\varphi \in C^\infty((0,T)\times\mathbb{R})$ be such that $V - \varphi$ has global maximum at (t,x), i.e.,

$$V(s,y) - V(t,x) \le \varphi(s,y) - \varphi(t,x)$$

for all $(s,y) \in [0,T]\times\mathbb{R}$. Then,

$$\begin{aligned} 0 &= \inf_a \mathbb{E}\left[\int_t^\tau l(X_s,a_s)\mathrm{d}s + V(\tau,X_\tau) - V(t,x)\right] \\ &\le \inf_a \mathbb{E}\left[\int_t^\tau l(X_s,a_s)\mathrm{d}s + \varphi(\tau,X_\tau) - \varphi(t,x)\right]. \end{aligned}$$

Thus,

$$\varphi_t(t,x) + \inf_a \left\{ l(x,a) + \varphi_x(t,x)b(x,a) + \frac{1}{2}\varphi_{xx}(t,x)\sigma^2(x,a) \right\} \ge 0.$$

Viscosity Solutions – Definition

Consider the HJB equation

$$\begin{cases} V_t(t,x) + \inf_a \left\{ l(x,a) + V_x(t,x)b(x,a) + \frac{1}{2} V_{xx}(t,x)\sigma^2(x,a) \right\} = 0 \\ V(T,x) = g(x). \end{cases} \tag{1}$$

Definition (Viscosity Solution)

$V \in C([0,T] \times \mathbb{R})$ *is viscosity subsolution of* (1), *if*

- $V(T,x) \leq g(x), \quad x \in \mathbb{R};$
- $\forall \varphi \in C^\infty((0,T) \times \mathbb{R})$ *such that* $V - \varphi$ *has a global maximum at* (t,x), *it holds*

$$\varphi_t(t,x) + \inf_a \left\{ l(x,a) + \varphi_x(t,x)b(x,a) + \frac{1}{2}\varphi_{xx}(t,x)\sigma^2(x,a) \right\} \geq 0$$

Viscosity Solutions – Consistency

Let $V \in C^{1,2}((0,T) \times \mathbb{R})$ be a classical solution of

$$\begin{cases} V_t(t,x) + \inf_a \left\{ l(x,a) + V_x(t,x)b(x,a) + \frac{1}{2} V_{xx}(t,x)\sigma^2(x,a) \right\} = 0 \\ V(T,x) = g(x) \end{cases}$$

and let $\varphi \in C^\infty((0,T) \times \mathbb{R})$ be such that $V - \varphi$ has a global maximum at (t,x). Then

$$V_t(t,x) = \varphi_t(t,x), \quad V_x(t,x) = \varphi_x(t,x), \quad V_{xx}(t,x) \leq \varphi_{xx}(t,x).$$

Hence,

$$\begin{aligned} &\varphi_t(t,x) + \inf_a \left\{ l(x,a) + \varphi_x(t,x)b(x,a) + \frac{1}{2}\varphi_{xx}(t,x)\sigma^2(x,a) \right\} \\ &\geq V_t(t,x) + \inf_a \left\{ l(x,a) + V_x(t,x)b(x,a) + \frac{1}{2} V_{xx}(t,x)\sigma^2(x,a) \right\} = 0 \end{aligned}$$

$\implies$ Every classical solution is a viscosity solution.

Remark: Proof of Uniqueness is more involved.

Viscosity Differentials

If $v \in C^{1,2}([0,T] \times \mathbb{R})$, it holds

$$\lim_{\tau \downarrow t, z \to x} \frac{1}{|\tau - t| + |z - x|^2} \Big[v(\tau, z) - v(t,x) - \partial_t v(t,x)(\tau - t) - \langle Dv(t,x), z - x \rangle - \frac{1}{2} \langle z - x, D^2 v(t,x)(z - x) \rangle \Big] = 0.$$

Weaker notion of differentiability:

Definition (Viscosity Superdifferential)

We say $(G, p, P) \in D^{1,2,+}_{t+,x} v(t,x)$ *if*

$$\limsup_{\tau \downarrow t, z \to x} \frac{1}{|\tau - t| + |z - x|^2} \Big[v(\tau, z) - v(t,x) - G(\tau - t) - \langle p, z - x \rangle - \frac{1}{2} \langle z - x, P(z - x) \rangle \Big] \leq 0.$$

Viscosity Solutions – Alternative Definition

It holds:

$$\boxed{(G, p, P) \in D^{1,2,+}_{t,x} v(t, x)}$$

$$\Longleftrightarrow$$

$\exists \phi \in C^{1,2}((0, T) \times \mathbb{R})$ such that:

1. $v - \phi$ attains maximum at (t, x),
2. $(\phi(t, x), \partial_t \phi(t, x), D\phi(t, x), D^2\phi(t, x)) = (v(t, x), G, p, P)$.

Definition (Viscosity Solution II)

$V \in C([0, T] \times \mathbb{R})$ *is viscosity subsolution of* (1), *if*

- $V(T, x) \leq g(x), \quad x \in \mathbb{R}$;
- *for every* $(G, p, P) \in D^{1,2,+}_{t,x} V(t, x)$

$$G + \inf_a \left\{ l(x, a) + pb(x, a) + \frac{1}{2} P\sigma^2(x, a) \right\} \geq 0.$$

Verification Theorem – Smooth Version

Theorem (Verification Theorem)

Assume V is sufficiently regular. If

$$a^*(s) \in \arg\min_a \Big\{ l(X^*(s), a) + \langle DV(s, X^*(s)), b(X^*(s), a)\rangle + \frac{1}{2} Tr[\sigma(X^*(s), a)\sigma^*(X^*(s), a) D^2 V(s, x)] \Big\} \quad (\star)$$

then a^* is optimal.

Due to HJB equation, $(\star)$ holds iff

$$V_t(s, X^*(s)) + \langle AX^*(s), DV(s, X^*(s))\rangle + \mathcal{H}(X^*(s), a^*(s), DV(s, X^*(s)), D^2 V(s, X^*(s))) = 0,$$

where

$$\mathcal{H}(x, a, p, P) := l(x, a) + \langle p, b(x, a)\rangle + \frac{1}{2}\mathrm{Tr}\left[\sigma(x, a)\sigma^*(x, a) P\right].$$

Verification Theorem – Nonsmooth Version

Theorem (Stannat, W. (to appear in Ann. Appl. Probab. 2024+))

Assume

- $\|\sigma(x,a)\|_{L_2(\Xi, H_0^1(\mathcal{O}))} \leq C(1 + \|x\|_{H_0^1(\mathcal{O})})$
- $V(t+\tau, x) - V(t,x) \leq C(1 + \|x\|^2_{H_0^1(\mathcal{O})})\tau$
- $V(t,\cdot) - C\|\cdot\|^2_{L^2(\mathcal{O})}$ *is concave.*

Let (X^*, a^*) *be an admissible pair. Suppose there are adapted processes* (G, p, P) *taking values in* $\mathbb{R}$, $H_0^1(\mathcal{O})$ *and* $L_2(L^2(\mathcal{O}))$, *such that for almost all* $s \in [t, T]$*:*

$$(G_s, p_s, P_s) \in D^{1,2,+}_{s+,x} V(s, X^*(s))$$

$\mathbb{P}$*-almost surely, and*

$$\mathbb{E}\left[\int_t^T G_s + \langle \Delta X^*(s), p_s\rangle_{H^{-1}(\mathcal{O})\times H_0^1(\mathcal{O})} + \mathcal{H}(X^*(s), a^*(s), p_s, P_s)\mathrm{d}s\right] \geq 0.$$

Then (X^*, a^*) *is an optimal pair.*

Some Comments about the Proof

Finite dimensional case:

- Using $(G_s, p_s, P_s) \in D^{1,2,+}_{s+,x} V(s, X^*(s))$, construct a test function ϕ.
- Use the fact that V is a viscosity solution.

Infinite dimensional case:

- We need to make sense of

$$\langle \Delta x, D\phi(s,x) \rangle \quad x \in L^2(\mathcal{O}).$$

  $\rightsquigarrow$ Need to restrict class of test functions.

- Our solution: Use higher regularity of $X(s) \in H^1_0(\mathcal{O})$. Make sense of

$$\langle \Delta X(s), D\phi(s, X(s)) \rangle_{H^{-1}(\mathcal{O}) \times H^1_0(\mathcal{O})}.$$

Optimal Synthesis

Recall:

Remark

Assume $l(x, a) = l_1(x) + l_2(a)$. Then, under certain assumptions,

$$a^*(s) = Dl_2^{-1}(DV(s, X^*(s))).$$

is optimal.

Problem: Viscosity solution is only continuous.

Next steps:

1. Prove higher regularity of the value function.
2. Construct optimal feedbacks.

Higher Regularity of the Value Function

For step 1:

Theorem (Lasry, Lions 1986)

Let $v : H \to \mathbb{R}$ be semiconvex and semiconcave. Then $v \in C^{1,1}(H)$.

Definition

$v : H \to \mathbb{R}$ is semiconcave if

$$\lambda v(x) + (1-\lambda)v(x') - v(\lambda x + (1-\lambda)x') \leq C\lambda(1-\lambda)\|x - x'\|_H^2$$

for all $\lambda \in [0,1]$ and $x, x' \in H$.

Remark

v semiconcave if and only if $x \mapsto v(x) - C\|x\|^2$ concave for some C.

Semiconcavity of the Value Function

Theorem (de Feo, Święch, W. (2023+))

Let

- b, σ, l, g *be Lipschitz in* x *and linearly growing in* (x, a),
- b, σ *be* $C^{1,1}$ *in* x,
- l, g *be semiconcave in* x.

Then, for every $t \in [0, T]$, *the function* $V(t, \cdot)$ *is semiconcave.*

Proof (Sketch).

Use stochastic representation

$$V(t,x) = \inf_{a(\cdot)} \mathbb{E}\left[\int_t^T l(X(s), a(s))\mathrm{d}s + g(X(T))\right]$$

and standard estimates for SDEs. □

Semiconvexity of the Value Function

Theorem (de Feo, Święch, W. (2023+))

Let

- σ *be independent of* a
- *assume mild regularity assumptions on the coefficients*
- g *be semiconvex*
- $H \times \Lambda \ni (x, a) \mapsto l(x, a) + C\|x\|_H^2 - \nu\|a\|_\Lambda^2$ *be convex*

Then there is a constant ν_0 *such that if* $\nu \geq \nu_0$*, then* $V(t, \cdot)$ *is semiconvex.*

Theorem (de Feo, Święch, W. (2023+))

Let $H = L^2(\mathcal{O})$ *and let*

- σ *be independent of* (x, a)*,* b *be of Nemytskii type and convex*
- e^{sA} *be positivity preserving*
- l, g *convex and nonincreasing in* x*.*

Then $V(t, \cdot)$ *is convex.*

Optimal Synthesis – Nonsmooth Version

Assumption

Let there be a Lipschitz continuous selection function

$$\gamma : H \times H \to \Lambda_0, \quad (x,p) \mapsto \gamma(x,p) \in \arg\min_a \{\langle p, b(x,a)\rangle + l(x,a)\}.$$

Theorem (de Feo, Święch, W. (2023+))

Let $V(t,\cdot) \in C^{1,1}(H)$ *and let* σ *be independent of the control. Then the pair* $(a^*(s), X^*(s))$*, where*

$$\begin{cases} a^*(s) = \gamma(X^*(s), DV(s, X^*(s))) \\ X^*(s) = X(s,t,x;a^*(\cdot)) \end{cases}$$

is an optimal couple.

Optimal Synthesis – Proof

Proof.

Consider the linear equation

$$v_t + \langle Ax, Dv\rangle_H + \frac{1}{2}\mathrm{Tr}[\sigma(x)\sigma^*(x)D^2 v] + \langle \tilde{b}(t,x), Dv\rangle_H + \tilde{l}(t,x) = 0,$$

where $\tilde{b}(t,x) := b(x,\gamma(x,DV(t,x)))$ and $\tilde{l}(t,x) := l(x,\gamma(x,DV(t,x)))$.
This equation has unique viscosity solution which is given by

$$v(t,x) = \mathbb{E}\left[\int_t^T \tilde{l}(s,X(s))\mathrm{d}s + g(X(T))\right],$$

where $X(s)$ is the solution of

$$\begin{cases} \mathrm{d}X(s) = [AX(s) + \tilde{b}(s,X(s))]\mathrm{d}s + \sigma(X(s))\mathrm{d}W(s) \\ X(t) = x. \end{cases}$$

□

References

F. de Feo, A. Święch and L. Wessels

Stochastic optimal control in Hilbert spaces: $C^{1,1}$ regularity of the value function and optimal synthesis via viscosity solutions

Submitted, arXiv:2310.03181

W. Stannat and L. Wessels

Necessary and sufficient conditions for optimal control of semilinear stochastic partial differential equations

to appear in Ann. Appl. Probab., arXiv:2112.09639