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Stochastic optimal control in Hilbert spaces: Optimal synthesis via viscosity solutions

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Setting

Control Problem

Cost functional:

$$J(t,x;a(\cdot)) := \mathbb{E}\left[\int_t^T l(X(s),a(s))\mathrm{d}s + g(X(T))\right]$$

State:

$$\begin{cases} \mathrm{d}X(s) = [AX(s) + b(X(s),a(s))]\mathrm{d}s + \sigma(X(s),a(s))\mathrm{d}W(s), & s \in [t,T] \\ X(t) = x \in H \end{cases}$$

Here:

- $a(\cdot) : [t,T] \times \Omega \to \Lambda$ control
- $l : H \times \Lambda \to \mathbb{R}$ and $g : H \to \mathbb{R}$ running and terminal cost
- $A : \mathcal{D}(A) \subset H \to H$ linear, unbounded operator
- $b : H \times \Lambda \to H$ and $\sigma : H \times \Lambda \to L_2(\Xi, H)$ drift and noise coefficient
- $(W(s))_{s\in[t,T]}$ cylindrical Wiener process

Value Function

$$V(t,x) := \inf_{a(\cdot)} J(t,x;a(\cdot)),$$

Hamilton–Jacobi–Bellman Equation

The value function is a viscosity solution of

$$\begin{cases} v_t + \langle Ax, Dv\rangle_H + \inf_{a\in\Lambda_0} \mathcal{F}(x,Dv,D^2v,a) = 0, & (t,x) \in (0,T) \times H \\ v(T,\cdot) = g, \end{cases}$$

where the Hamiltonian function $\mathcal{F} : H \times H \times S(H) \times \Lambda_0 \to \mathbb{R}$ is given by

$$\mathcal{F}(x,p,P,a) := \frac{1}{2}\mathrm{Tr}\,[\sigma(x,a)\sigma^*(x,a)P] + \langle b(x,a), p\rangle_H + l(x,a).$$

Optimal Synthesis under Smoothness Assumptions

Classical Result

Assume that V is smooth and $l(x,a) = l_1(x) + l_2(a)$. Then, under certain assumptions,

$$a^*(s) = Dl_2^{-1}(DV(s,X^*(s)))$$

is optimal.

Problem

Value function is not differentiable in general.

Goal

1. Prove higher regularity of the value function
2. Construct optimal feedbacks

Regularity of the Value Function

Lasry, Lions (1986)

Let $v : H \to \mathbb{R}$ be semiconvex and semiconcave. Then $v \in C^{1,1}(H)$.

Semiconcavity

Let

$$\begin{cases} b,\sigma,l,g \text{ be Lipschitz in } x \text{ and linearly growing in } (x,a) \\ b,\sigma \text{ be } C^{1,1} \text{ in } x \\ l,g \text{ be semiconcave in } x. \end{cases} \tag{A0}$$

Then, for every $t \in [0,T]$, the function $V(t,\cdot)$ is semiconcave.

Semiconvexity

Case 1

Let

$$\begin{cases} \sigma \text{ be independent of } a \\ \text{assume mild regularity assumptions on the coefficients} \\ g \text{ be semiconvex} \\ H \times \Lambda \ni (x,a) \mapsto l(x,a) + C\|x\|_H^2 - \nu\|a\|_\Lambda^2 \text{ be convex} \end{cases} \tag{A1}$$

Then there is a constant ν_0 such that if $\nu \geq \nu_0$, then $V(t,\cdot)$ is semiconvex.

Case 2

Let

$$\begin{cases} b : H \times \Lambda \to H \text{ and } \sigma : H \times \Lambda \to L_2(\Xi,H) \text{ be bounded and linear} \\ l : H \times \Lambda_0 \to \mathbb{R} \text{ and } g : H \to \mathbb{R} \text{ be convex.} \end{cases} \tag{A2}$$

Then, $V(t,\cdot)$ is convex.

Case 3

Let $H = L^2(\mathcal{O})$ and let

$$\begin{cases} \sigma \text{ be independent of } (x,a), b \text{ be of Nemytskii type and convex} \\ e^{sA} \text{ be positivity preserving} \\ l,g \text{ convex and nonincreasing in } x. \end{cases} \tag{A3}$$

Then $V(t,\cdot)$ is convex.

$C^{1,1}$ Regularity of the Value Function

Theorem

Let Assumption (A0) as well as one of the assumptions (A1), (A2), or (A3) be satisfied. Then $V(t,\cdot) \in C^{1,1}(H)$ for every $t \in [0,T]$.

Optimal Synthesis

Main Assumptions

- Let $V(t,\cdot) \in C^{1,1}(H)$.
- Let $\sigma(x,a) \equiv \sigma(x)$ be independent of the control.
- There exists a selection function

$$\gamma : H \times H \to \Lambda_0, \quad (x,p) \mapsto \gamma(x,p) \in \Gamma(x,p),$$

which is Lipschitz continuous in both variables.

Theorem

Let the assumptions above be satisfied. Then, the pair $(a^*(s), X^*(s))$, where

$$\begin{cases} a^*(s) = \gamma(X^*(s), DV(s,X^*(s))) \\ X^*(s) = X(s,t,x;a^*(\cdot)) \end{cases}$$

is an optimal couple.

Example

Controlled SPDE

Let

$$\begin{cases} A := \sum_{i,j=1}^d \partial_i(a_{ij}\partial_j) + \sum_{i=1}^d b_i\partial_i + c \\ \mathcal{D}(A) = H_0^1(\mathcal{O}) \cap H^2(\mathcal{O}) \end{cases}$$

where $a_{ij} = a_{ji}, b_i \in W^{1,\infty}(\mathcal{O}), i,j = 1,\dots,d$, $c \in L^\infty(\mathcal{O})$, and there is a constant $\theta > 0$ such that

$$\sum_{i,j=1}^d a_{ij}\xi_i\xi_j \geq \theta|\xi|^2$$

Cost functional:

$$J(t,x;a(\cdot)) := \mathbb{E}\left[\int_t^T \int_{\mathcal{O}} (l_1(X(s,\xi)) + l_2(a(s,\xi)))\,\mathrm{d}\xi\mathrm{d}s + \int_{\mathcal{O}} g(X(T,\xi))\mathrm{d}\xi\right]$$

State:

$$\begin{cases} \mathrm{d}X(s) = [AX(s) + \mathfrak{b}(X(s)) - a(s)]\mathrm{d}s + \sigma\mathrm{d}W(s), & s \in [t,T] \\ X(t) = x \in L^2(\mathcal{O}). \end{cases}$$

Optimal feedback control:

$$a^*(s) = Dl_2^{-1}(DV(s,X^*(s))).$$

References

[1] F. De Feo, A. Święch and L. Wessels, *Stochastic optimal control in Hilbert spaces: $C^{1,1}$ regularity of the value function and optimal synthesis via viscosity solutions*, submitted, arxiv:2310.03181.

[2] J.-M. Lasry and P.-L. Lions, *A remark on regularization in Hilbert spaces*, Israel J. Math. 55 (1986), 257–266.