Stochastic optimal control in Hilbert spaces: Optimal synthesis via viscosity solutions

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Seminar on Stochastic Processes
Rice University, March 13 – 16, 2024

Georgia Institute of Technology

Optimal Control via Dynamic Programming

Minimize

$$J(t, x; a(\cdot)) := \mathbb{E}\left[\int_t^T l(X(s), a(s))\mathrm{d}s + g(X(T))\right]$$

over admissible controls $a(\cdot) : [t, T] \times \Omega \to \Lambda$ subject to

$$\begin{cases} \mathrm{d}X(s) = [AX(s) + b(X(s), a(s))]\mathrm{d}s + \sigma(X(s), a(s))\mathrm{d}W(s), & s \in [t, T] \\ X(t) = x \in H, \end{cases}$$

where

- $l : H \times \Lambda \to \mathbb{R}$ and $g : H \to \mathbb{R}$ are running and terminal cost
- $A : \mathcal{D}(A) \subset H \to H$ linear unbounded operator
- b and σ drift and diffusion coefficient
- $(W(s))_{s\in[t,T]}$ cylindrical Wiener process

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Value Function:

$$V(t,x) := \inf_{a(\cdot)} J(t,x;a(\cdot)).$$

Dynamic Programming Principle:

$$V(t,x) = \inf_{a(\cdot)} \mathbb{E}\left[\int_t^\tau l(X(s),a(s))\mathrm{d}s + V(\tau, X(\tau))\right], \qquad \forall \tau \in [t,T].$$

HJB Equation and Optimal Synthesis

Hamilton–Jacobi–Bellman equation:

$$\begin{cases} V_t(t,x) + \langle Ax, DV(t,x)\rangle \\ \quad + \inf_a \left\{ \frac{1}{2}\mathrm{Tr}[\sigma(x,a)\sigma^*(x,a)D^2V(t,x)] + \langle DV(t,x), b(x,a)\rangle + l(x,a) \right\} = 0 \\ V(T,x) = g(x). \end{cases}$$

Optimal Synthesis: Assume V is smooth and $l(x,a) = l_1(x) + l_2(a)$. Then,

$$a^*(s) = Dl_2^{-1}(DV(s, X^*(s)))$$

is optimal.

Question: What can we do if V is not smooth?

$\rightarrow$ For the answer, come and see my poster!