Finite Dimensional Projections of HJB Equations in the Wasserstein Space

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Finite Dimensional Control Problem

Fix $0 \leq t < T < \infty$ and $\mathbf{x} = (x_1, \dots, x_n) \in (\mathbb{R}^d)^n$. Minimize

$$J_n(t, \mathbf{x}; \mathbf{a}(\cdot)) := \mathbb{E}\left[\int_t^T \left(\frac{1}{n}\sum_{i=1}^n \left(l_1(X_i(s), \mu_{\mathbf{X}(s)}) + l_2(a_i(s))\right)\right) \mathrm{d}s + \mathcal{U}_T(\mu_{\mathbf{X}(T)})\right]$$

subject to

$$\begin{cases} \mathrm{d}X_i(s) = [-a_i(s) + b(X_i(s), \mu_{\mathbf{X}(s)})]\mathrm{d}s + \sigma(X_i(s), \mu_{\mathbf{X}(s)})\mathrm{d}W(s) \\ X_i(t) = x_i \in \mathbb{R}^d, \end{cases}$$

$i = 1, \dots, n$. Here:

- b, σ drift and noise coefficient, respectively.
- $(W(s))_{s \in [t,T]}$ Wiener process – common noise.
- $\mu_{\mathbf{x}} := \frac{1}{n}\sum_{i=1}^n \delta_{x_i}$ empirical measure
- $\mathbf{a}(\cdot) = (a_1(\cdot), \dots, a_n(\cdot)) : [t, T] \times \Omega' \to (\mathbb{R}^d)^n$ control
- $l_1 : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ and $l_2 : \mathbb{R}^d \to \mathbb{R}$ running cost
- $\mathcal{U}_T : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ terminal cost

Example for $\mathcal{U}_T$

$\mathcal{U}_T : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$.

1. $\phi : \mathbb{R}^d \to \mathbb{R}$:

$$\mathcal{U}_T(\mu) := \int_{\mathbb{R}^d} \phi(x)\mu(\mathrm{d}x)$$

2. $\phi : \mathbb{R}^d \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$:

$$\mathcal{U}_T(\mu) := g\left(\int_{\mathbb{R}^d} \phi(x)\mu(\mathrm{d}x)\right)$$

3. $\eta : (\mathbb{R}^d)^m \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$:

$$\mathcal{U}_T(\mu) = g\left(\int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \eta(x_1, \dots, x_m)\mu(\mathrm{d}x_1) \cdots \mu(\mathrm{d}x_m)\right)$$

Dynamic Programming

Define **value function**

$$u_n(t,\mathbf{x}) := \inf_{\mathbf{a}(\cdot)} J_n(t,\mathbf{x};\mathbf{a}(\cdot)).$$

Satisfies **Hamilton–Jacobi–Bellman equation**

$$\begin{cases} \partial_t u_n + \frac{1}{2}\mathrm{Tr}(A_n(\mathbf{x},\mu_{\mathbf{x}})D^2 u_n) - \frac{1}{n}\sum_{i=1}^n H(x_i,\mu_{\mathbf{x}}, nD_{x_i}u_n) = 0 \\ u_n(T,\mathbf{x}) = \mathcal{U}_T(\mu_{\mathbf{x}}), \quad \mathbf{x}\in(\mathbb{R}^d)^n, \end{cases} \tag{HJB$_n$}$$

where $A_n(\mathbf{x},\mu)$ is $nd\times nd$-square matrix, Hamiltonian $H:\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)\times\mathbb{R}^d\to\mathbb{R}$ is given by

$$H(x,\mu,p) = -b(x,\mu)\cdot p - l_1(x,\mu) + \sup_{q\in\mathbb{R}^d}(q\cdot p - l_2(q)).$$

Example: $n=2$, $d=1$, $\sigma\equiv 1$. Then

$$A_2(x,\mu_{\mathbf{x}}) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

$\rightsquigarrow$ degenerate!

Limiting Equation

Formally u_n converges to solution of

$$\begin{cases} \partial_t \mathcal{U}(t,\mu) + \frac{1}{2}\int_{\mathbb{R}^d} \operatorname{Tr}\left[D_x \partial_\mu \mathcal{U}(t,\mu)(x)\sigma(x,\mu)\sigma^\top(x,\mu)\right]\mu(\mathrm{d}x) \\ \quad + \frac{1}{2}\int_{(\mathbb{R}^d)^2} \operatorname{Tr}\left[\partial_\mu^2 \mathcal{U}(t,\mu)(x,x')\sigma^\top(x,\mu)\sigma(x',\mu)\right]\mu(\mathrm{d}x)\mu(\mathrm{d}x') \\ \quad - \int_{\mathbb{R}^d} H(x,\mu,\partial_\mu \mathcal{U}(t,\mu)(x))\mu(\mathrm{d}x) = 0 \\ \mathcal{U}(T,\mu) = \mathcal{U}_T(\mu), \quad \mu \in \mathcal{P}_2(\mathbb{R}^d) \end{cases} \qquad (\mathrm{HJB}_\infty)$$

PDE on Wasserstein space $\mathcal{P}_2(\mathbb{R}^d) \rightsquigarrow$ hard to solve.

Lions' lifting:

$$\begin{aligned} &\mathcal{P}_2(\mathbb{R}^d) && E := L^2(\Omega;\mathbb{R}^d), \quad \Omega = (0,1) \\ &\mathcal{V}: \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R} && V: E \to \mathbb{R}, \quad V(X) := \mathcal{V}(X_\# \mathcal{L}^1) \end{aligned}$$

Lifted equation:

$$\begin{cases} \partial_t V + \frac{1}{2}\sum_{m=1}^{d'} \langle D^2 V \Sigma(X) e_m', \Sigma(X) e_m' \rangle_E - \tilde{H}(X, X_\# \mathcal{L}^1, DV) = 0 \\ V(T,X) = U_T(X), \quad X \in E, \end{cases} \qquad (\mathrm{HJB}_\uparrow)$$

where U_T is lift of $\mathcal{U}_T$ and $(e_m')_{m=1,\dots,d'}$ standard basis of $\mathbb{R}^{d'}$.

Infinite Dimensional Control Problem

Consider infinite dimensional SDE

$$\begin{cases} \mathrm{d}X(s) = [-a(s) + B(X(s))]\mathrm{d}s + \Sigma(X(s))\mathrm{d}W(s) \\ X(t) = X \in E, \end{cases}$$

where

$$B : E \to E \qquad B(X)(\omega) := b(X(\omega), X_\# \mathcal{L}^1)$$

$$\Sigma : E \to L_2(\mathbb{R}^{d'}, E) \qquad \Sigma(X)(\omega) := \sigma(X(\omega), X_\# \mathcal{L}^1)$$

Cost functional

$$J(t, X; a(\cdot)) = \mathbb{E}\left[\int_t^T (L_1(X(s)) + L_2(a(s)))\,\mathrm{d}s + U_T(X(T))\right],$$

where

$$L_1(X) := \int_\Omega l_1(X(\omega), X_\# \mathcal{L}^1)\mathrm{d}\omega, \quad L_2(X) := \int_\Omega l_2(X(\omega))\mathrm{d}\omega.$$

Define value function

$$U(t, X) := \inf_{a(\cdot)\in\mathcal{A}_t} J(t, X, a(\cdot)).$$

Assumptions

Assumption (Regularity)

Let $l_1, \mathcal{U}_T, b, \sigma$ *be Lipschitz w.r.t.* $|\cdot| \times d_r(\cdot,\cdot)$*-distance,* $r \in [1,2)$*, and let their lifts be* $C^{1,1}$*.*

Assumption (Convexity of l_2)

Let

$$p \mapsto l_2(p) - \nu |p|^2$$

be convex for some constant $\nu \geq 0$*.*

Assumption (Linear-Convex Case)

Let the lifts of b *and* σ *be affine linear, and let the lifts of* l_1 *and* $\mathcal{U}_T$ *be convex.*

Convergence – Outline

1. Define $\tilde{\mathcal{V}}_n(t,\mu_{\mathbf{x}}) := u_n(t,\mathbf{x})$, $\quad \mu_{\mathbf{x}} = \frac{1}{n}\sum_{i=1}^n \delta_{x_i}$.
2. Show that $|\tilde{\mathcal{V}}_n(t,\mu_{\mathbf{x}}) - \tilde{\mathcal{V}}_n(s,\mu_{\mathbf{y}})| \le C d_r(\mu_{\mathbf{x}},\mu_{\mathbf{y}}) + C_R|t-s|^{\frac{1}{2}}$.
3. Extend $\tilde{\mathcal{V}}_n$ to $\mathcal{V}_n : [0,T]\times\mathcal{P}_2(\mathbb{R}^d)\to\mathbb{R}$ by

$$\mathcal{V}_n(t,\mu) := \sup\left\{\tilde{\mathcal{V}}_n(t,\beta) - 2Cd_r(\mu,\beta) : \beta = \frac{1}{n}\sum_{i=1}^n \delta_{x_i}\right\}.$$

4. Show that $|\mathcal{V}_n(t,\mu) - \mathcal{V}_n(s,\beta)| \le 2Cd_r(\mu,\beta) + C|t-s|^{\frac{1}{2}}$.
5. $\mathcal{P}_2(\mathbb{R}^d)$ is compactly embedded in $\mathcal{P}_r(\mathbb{R}^d)$, $r\in[1,2)$. Thus, we can apply Arzelà–Ascoli. $\implies$ $\mathcal{V}_n \to \mathcal{V}$ on bounded subsets of $\mathcal{P}_2(\mathbb{R}^d)$.

Define $V : [0,T]\times E \to \mathbb{R}$ by

$$V(t,X) := \mathcal{V}(t, X_{\#}\mathcal{L}^1).$$

Convergence – Theorem

Theorem

For every bounded set B in $\mathcal{P}_2(\mathbb{R}^d)$, we have

$$\lim_{n\to\infty} \sup \Bigg\{ \left| u_n(t,x_1,\dots,x_n) - \mathcal{V}\left(t, \frac{1}{n}\sum_{i=1}^n \delta_{x_i}\right) \right| \\ : (t,x_1,\dots,x_n) \in (0,T] \times (\mathbb{R}^d)^n, \frac{1}{n}\sum_{i=1}^n \delta_{x_i} \in B \Bigg\} = 0$$

and $\mathcal{V}$ is the unique L-viscosity solution of (HJB$_\infty$). Moreover, $V = U$, where U is the value function of the infinite dimensional control problem.

$C^{1,1}$ Regularity

Theorem (Lasry, Lions (1986))

$v : H \to \mathbb{R}$ *semiconcave, semiconvex, continuous* $\implies v \in C^{1,1}(H)$.

Proposition (Semiconcavity)

Let all coefficients be Lipschitz w.r.t. the $d_1(\cdot,\cdot)$-distance. Then, for every t, $U(t,\cdot)$ is semiconcave with the semiconcavity constant independent of t.

Proposition (Semiconvexity: Case 1)

Let all coefficients be Lipschitz w.r.t. the $d_1(\cdot,\cdot)$-distance. There is a constant ν_0 such that if ν in Assumption (Convexity of l_2) satisfies $\nu \geq \nu_0$, then, for every t, $U(t,\cdot)$ is semiconvex with the semiconvexity constant independent of t.

Proposition (Semiconvexity: Case 2)

Let the state equation be affine linear and the cost be convex. Then, for every t, $U(t,\cdot)$ is convex.

Projection

For $n \in \mathbb{N}$, $i \in \{1, \ldots, n\}$, let $A_i^n = \left(\frac{i-1}{n}, \frac{i}{n}\right) \subset \Omega$. Now, we introduce $V_n : [0,T] \times (\mathbb{R}^d)^n \to \mathbb{R}$,

$$V_n(t, x_1, \ldots, x_n) := V\left(t, \sum_{i=1}^{n} x_i \mathbf{1}_{A_i^n}\right).$$

Theorem (Projection under $C^{1,1}$ regularity)

Let $V(t,\cdot) \in C^{1,1}(E)$, for every t. Then, it holds

$$V_n(t, \mathbf{x}) = u_n(t, \mathbf{x}), \quad (t, \mathbf{x}) \in [0,T] \times (\mathbb{R}^d)^n.$$

Theorem (Projection without $C^{1,1}$ regularity)

Let the state equation be affine linear and let the coefficients of the cost functional be Lipschitz. Then, it holds

$$V_n(t, \mathbf{x}) = u_n(t, \mathbf{x}), \quad (t, \mathbf{x}) \in [0,T] \times (\mathbb{R}^d)^n.$$

Proof of Projection: $V_n \leq u_n$

Lemma

For any control $\mathbf{a}(\cdot) \in \mathcal{A}_t^n$, *we have*

$$J_n(t, \mathbf{x}; \mathbf{a}(\cdot)) = J(t, X_n^{\mathbf{x}}; a_n(\cdot)),$$

where $X_n^{\mathbf{x}} = \sum_{i=1}^n x_i \mathbf{1}_{A_i^n}$ *and* $a^n(\cdot) = \sum_{i=1}^n a_i(\cdot)\mathbf{1}_{A_i^n}$.

Proof.

Let $X(\cdot)$ be solution with initial condition $X_n^{\mathbf{x}}$ and control $a^n(\cdot)$. Then,

$$X(s) = \sum_{i=1}^n x_i \mathbf{1}_{A_i^n} - \int_t^s \sum_{i=1}^n a_i(r)\mathbf{1}_{A_i^n} \mathrm{d}r + \int_t^s B(X(r))\mathrm{d}r + \int_t^s \Sigma(X(r))\mathrm{d}W(r)$$

$\mathbb{P}$-a.s., for every $s \in [t, T]$.

Proof. (cont.)

Therefore, for almost every $\omega \in A_i^n = (\frac{i-1}{n}, \frac{i}{n})$, we have

$$X(s,\omega) = x_i - \int_t^s a_i(r)\mathrm{d}r + \int_t^s b(X(r,\omega), X(r)_\# \mathcal{L}^1)\mathrm{d}r + \int_t^s \sigma(X(r,\omega), X(r)_\# \mathcal{L}^1)\mathrm{d}W(r).$$

$\mu_{\mathbf{X}(s)} = (X_n^{\mathbf{X}(s)})_\# \mathcal{L}^1 \implies X(s,\omega) = X_n^{\mathbf{X}(s)}(\omega)$. Thus,

$$\begin{aligned} &J(t, X_n^{\mathbf{x}}; a^n(\cdot)) \\ &= \mathbb{E}\left[\int_t^T \left(\int_\Omega l_1(X_n^{\mathbf{X}(s)}(\omega), (X_n^{\mathbf{X}(s)})_\# \mathcal{L}^1)\mathrm{d}\omega + \int_\Omega l_2(X_n^{\mathbf{X}(s)}(\omega))\mathrm{d}\omega \right)\mathrm{d}s + U_T(X_n^{\mathbf{X}(T)})\right] \\ &= \mathbb{E}\left[\int_t^T \left(\frac{1}{n}\sum_{i=1}^n \left(l_1(X_i(s), \mu_{\mathbf{X}(s)}) + l_2(X_i(s))\right)\right)\mathrm{d}s + \mathcal{U}_T(\mu_{\mathbf{X}(T)})\right] \\ &= J_n(t, \mathbf{x}; \mathbf{a}(\cdot)). \end{aligned}$$

□

Proof of Projection: $V_n \leq u_n$

Lemma

For any control $\mathbf{a}(\cdot) \in \mathcal{A}_t^n$*, we have*

$$J_n(t, \mathbf{x}; \mathbf{a}(\cdot)) = J(t, X_n^{\mathbf{x}}; a_n(\cdot)),$$

where $X_n^{\mathbf{x}} = \sum_{i=1}^n x_i \mathbf{1}_{A_i^n}$ *and* $a_n(\cdot) = \sum_{i=1}^n a_i(\cdot) \mathbf{1}_{A_i^n}$.

Proposition

It holds

$$V_n(t, \mathbf{x}) \leq u_n(t, \mathbf{x})$$

for all $(t, \mathbf{x}) \in [0, T] \times (\mathbb{R}^d)^n$.

Proof of Projection: $V_n \geq u_n$ using $C^{1,1}$ Regularity

Proposition

Let $V(t,\cdot) \in C^{1,1}(E)$, for every $t \in [0,T]$. Then, it holds

$$V_n(t,\mathbf{x}) \geq u_n(t,\mathbf{x})$$

for all $(t,\mathbf{x}) \in [0,T] \times (\mathbb{R}^d)^n$.

Strategy: Show that V_n is a viscosity supersolution of (HJB$_n$) and use comparison.

Definition (Viscosity Supersolution)

$V_n \in C([0,T] \times (\mathbb{R}^d)^n)$ *is a viscosity supersolution of (HJB$_n$), if* $V_n(T,\mu_{\mathbf{x}}) \geq \mathcal{U}_T(\mu_{\mathbf{x}})$ *and for all test functions* $\varphi \in C^{1,2}((0,T) \times (\mathbb{R}^d)^n)$ *such that* $V_n - \varphi$ *has a global minimum at* $(t_0, \mathbf{x}_0)$*, it holds*

$$\partial_t \varphi + \frac{1}{2} \mathrm{Tr}(A_n(\mathbf{x}_0, \mu_{\mathbf{x}_0}) D^2 \varphi) - \frac{1}{n} \sum_{i=1}^{n} H(x_{0i}, \mu_{\mathbf{x}_0}, n D_{x_i} \varphi) \leq 0.$$

Proof of Projection: $V_n \geq u_n$ using $C^{1,1}$ Regularity

Proof.

Let $\varphi : (0,T) \times (\mathbb{R}^d)^n \to \mathbb{R}$ be a test function, i.e. (w.l.o.g)

- $(V_n - \varphi)(t_0, \mathbf{0}) = 0$ is global minimum
- $\partial_t \varphi(t_0, \mathbf{0}) = 0$ and $D\varphi(t_0, \mathbf{0}) = \mathbf{0}$.

$V : [0,T] \times E \to \mathbb{R}$ is viscosity solution of HJB$_\uparrow$. $\rightsquigarrow$ lift test function to E.
Natural embedding of $(\mathbb{R}^d)^n$ in E:

$$(\mathbb{R}^d)^n \ni \mathbf{x} = (x_1, \dots, x_n) \quad \longleftrightarrow \quad \sum_{i=1}^{n} x_i \mathbf{1}_{A_i^n} \in E$$

Extend $\sqrt{n}\mathbf{1}_{A_i^n}$ to basis of $L^2(\Omega)$, denoted by $(f_i)_{i \in \mathbb{N}}$, and let $(e_k)_{k=1,\dots d}$ be basis of $\mathbb{R}^d$. Then

$$X = P_n X + P_n^\perp X = \sum_{i=1}^{n} y_i f_i + \sum_{i=n+1}^{\infty} y_i f_i = \sum_{i=1}^{n} \sum_{k=1}^{d} y_i^k f_i e_k + \sum_{i=n+1}^{\infty} \sum_{k=1}^{d} y_i^k f_i e_k.$$

Proof. (cont.)

Define $\tilde{\varphi} : (0,T) \times E \to \mathbb{R}$ by

$$\tilde{\varphi}(t,X) := \varphi(t, \sqrt{n}\mathbf{y}).$$

Consider test function

$$(0,T) \times E \ni (t,X) \mapsto V(t,X) - \tilde{\varphi}(t,X) + \varepsilon((t-t_0)^2 + \|P_n X\|_E^2) + \frac{\varepsilon}{\delta^2}\|P_n^\perp X\|_E^2.$$

Now, show that for (t,X) in the boundary of some neighborhood K_δ around $(t_0,0)$

$$V(t,X) - \tilde{\varphi}(t,X) + \varepsilon((t-t_0)^2 + \|P_n X\|_E^2) + \frac{\varepsilon}{\delta^2}\|P_n^\perp X\|_E^2 \geq \gamma > 0.$$

Proof. (cont.)

Since $V(t_0,0) - \tilde{\varphi}(t_0,0) = 0$, we can apply Ekeland–Lebourg to produce minimum of

$$V(t,X) - \tilde{\varphi}(t,X) + \varepsilon((t-t_0)^2 + \|P_n X\|_E^2) + \frac{\varepsilon}{\delta^2}\|P_n^\perp X\|_E^2 + at + \langle Z, X\rangle_E$$

at some point (t_δ, X_δ) inside K_δ. Since V is viscosity solution of $\text{HJB}_\uparrow$, we have

$$\begin{aligned}
&\partial_t \tilde{\varphi}(t_\delta, X_\delta) - 2\varepsilon(t_\delta - t_0) - a \\
&\quad + \frac{1}{2}\sum_{m=1}^{d'} \langle (D^2\tilde{\varphi}(t_\delta, X_\delta) - 2\varepsilon P_n - 2\frac{\varepsilon}{\delta^2}P_n^\perp)\Sigma(X_\delta)e'_m, \Sigma(X_\delta)e'_m\rangle_E \\
&\qquad - \tilde{H}(X_\delta, (X_\delta)_\# \mathcal{L}^1, D\tilde{\varphi}(t_\delta, X_\delta) - 2\varepsilon P_n X_\delta - 2\frac{\varepsilon}{\delta^2}P_n^\perp X_\delta - Z) \le 0.
\end{aligned}$$

Taking the limit $\varepsilon \to 0$ concludes the proof. □

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