

Finite Dimensional Projections of HJB Equations in the Wasserstein Space

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Finite Dimensional Control Problem

Fix $0 \leq t < T < \infty$ and $\mathbf{x} = (x_1, \dots, x_n) \in (\mathbb{R}^d)^n$. Minimize

$$J_n(t, \mathbf{x}; \mathbf{a}(\cdot)) := \mathbb{E} \left[\int_t^T \left(\frac{1}{n} \sum_{i=1}^n (l_1(X_i(s), \mu_{\mathbf{X}(s)}) + l_2(a_i(s))) \right) ds + \mathcal{U}_T(\mu_{\mathbf{X}(T)}) \right]$$

subject to

$$\begin{cases} dX_i(s) = [-a_i(s) + b(X_i(s), \mu_{\mathbf{X}(s)})]ds + \sigma(X_i(s), \mu_{\mathbf{X}(s)})dW(s) \\ X_i(t) = x_i \in \mathbb{R}^d, \end{cases}$$

$i = 1, \dots, n$. Here:

- $\mu_{\mathbf{x}} := \frac{1}{n} \sum_{i=1}^n \delta_{x_i}$ empirical measure
- $(W(s))_{s \in [t, T]}$ Wiener process – common noise.
- $\mathbf{a}(\cdot) = (a_1(\cdot), \dots, a_n(\cdot)) : [t, T] \times \Omega' \rightarrow (\mathbb{R}^d)^n$ control
- $l_1 : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \rightarrow \mathbb{R}$ and $l_2 : \mathbb{R}^d \rightarrow \mathbb{R}$ running cost
- $\mathcal{U}_T : \mathcal{P}_2(\mathbb{R}^d) \rightarrow \mathbb{R}$ terminal cost

Dynamic Programming

Define **value function**

$$u_n(t, \mathbf{x}) := \inf_{\mathbf{a}(\cdot)} J_n(t, \mathbf{x}; \mathbf{a}(\cdot)).$$

Satisfies **Hamilton–Jacobi–Bellman equation**

$$\begin{cases} \partial_t u_n + \frac{1}{2} \text{Tr}(A_n(\mathbf{x}, \mu_{\mathbf{x}}) D^2 u_n) - \frac{1}{n} \sum_{i=1}^n H(x_i, \mu_{\mathbf{x}}, n D_{x_i} u_n) = 0 \\ u_n(T, \mathbf{x}) = \mathcal{U}_T(\mu_{\mathbf{x}}), \quad \mathbf{x} \in (\mathbb{R}^d)^n, \end{cases} \quad (\text{HJB}_n)$$

where $A_n(\mathbf{x}, \mu)$ is $nd \times nd$ -square matrix, Hamiltonian

$H : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \rightarrow \mathbb{R}$ is given by

$$H(x, \mu, p) = -b(x, \mu) \cdot p - l_1(x, \mu) + \sup_{q \in \mathbb{R}^d} (q \cdot p - l_2(q)).$$

Example: $n = 2$, $d = 1$, $\sigma \equiv 1$. Then

$$A_2(\mathbf{x}, \mu_{\mathbf{x}}) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

↪ Degenerate!

Limiting Equation

Formally u_n converges to solution of

$$\begin{cases} \partial_t \mathcal{U}(t, \mu) + \frac{1}{2} \int_{\mathbb{R}^d} \text{Tr} [D_x \partial_\mu \mathcal{U}(t, \mu)(x) \sigma(x, \mu) \sigma^\top(x, \mu)] \mu(\mathrm{d}x) \\ \quad + \frac{1}{2} \int_{(\mathbb{R}^d)^2} \text{Tr} [\partial_\mu^2 \mathcal{U}(t, \mu)(x, x') \sigma(x, \mu) \sigma^\top(x', \mu)] \mu(\mathrm{d}x) \mu(\mathrm{d}x') \\ \quad - \int_{\mathbb{R}^d} H(x, \mu, \partial_\mu \mathcal{U}(t, \mu)(x)) \mu(\mathrm{d}x) = 0 \\ \mathcal{U}(T, \mu) = \mathcal{U}_T(\mu), \quad \mu \in \mathcal{P}_2(\mathbb{R}^d) \end{cases} \quad (\text{HJB}_\infty)$$

PDE on Wasserstein space $\mathcal{P}_2(\mathbb{R}^d) \rightsquigarrow$ hard to solve.

Lions' lift:

$$\begin{aligned} \mathcal{P}_2(\mathbb{R}^d) & & E &:= L^2(\Omega; \mathbb{R}^d), \quad \Omega = (0, 1) \\ \mathcal{V} : \mathcal{P}_2(\mathbb{R}^d) &\rightarrow \mathbb{R} & V : E &\rightarrow \mathbb{R}, \quad V(X) := \mathcal{V}(X_\# \mathcal{L}^1) \end{aligned}$$

Lifted equation:

$$\begin{cases} \partial_t V + \frac{1}{2} \text{Tr}(\Sigma(X)(\Sigma(X))^* D^2 V) - \tilde{H}(X, X_\# \mathcal{L}^1, DV) = 0 \\ V(T, X) = U_T(X), \quad X \in E, \end{cases} \quad (\text{HJB}_\uparrow)$$

where U_T is lift of $\mathcal{U}_T$.

Infinite Dimensional Control Problem

Consider infinite dimensional SDE

$$\begin{cases} dX(s) = [-a(s) + B(X(s))]ds + \Sigma(X(s))dW(s) \\ X(t) = X \in E, \end{cases}$$

where

$$B : E \rightarrow E$$

$$B(X)(\omega) := b(X(\omega), X_{\#}\mathcal{L}^1)$$

$$\Sigma : E \rightarrow L(\mathbb{R}^{d'}, E)$$

$$\Sigma(X)(\omega) := \sigma(X(\omega), X_{\#}\mathcal{L}^1)$$

Cost functional

$$J(t, X; a(\cdot)) = \mathbb{E} \left[\int_t^T (L_1(X(s)) + L_2(a(s))) ds + U_T(X(T)) \right],$$

where

$$L_1(X) := \int_{\Omega} l_1(X(\omega), X_{\#}\mathcal{L}^1) d\omega, \quad L_2(X) := \int_{\Omega} l_2(X(\omega)) d\omega.$$

Define value function

$$U(t, X) := \inf_{a(\cdot)} J(t, X, a(\cdot)).$$

Main Results

Objects of interest:

- $u_n : [0, T] \times (\mathbb{R}^d)^n \rightarrow \mathbb{R}$ – finite-dim value functions
- $U : [0, T] \times L^2(\Omega; \mathbb{R}^d) \rightarrow \mathbb{R}$ – infinite-dim value function

How to connect these two?

- Define $\tilde{\mathcal{V}}_n(t, \mu_{\mathbf{x}}) := u_n(t, \mathbf{x})$, $\mu_{\mathbf{x}} = \frac{1}{n} \sum_{i=1}^n \delta_{x_i}$.
- Extend $\tilde{\mathcal{V}}_n$ to a function $\mathcal{V}_n : [0, T] \times \mathcal{P}_2(\mathbb{R}^d) \rightarrow \mathbb{R}$

Main results:

- 1 Convergence of $\mathcal{V}_n \rightarrow \mathcal{V}$, where $\mathcal{V}$ lifts to U .
- 2 $C^{1,1}$ regularity of U .
- 3 Exact projection of U onto u_n .
- 4 Lifting and projection of optimal controls.

Convergence

- 1 Define $\tilde{V}_n(t, \mu_{\mathbf{x}}) := u_n(t, \mathbf{x})$, $\mu_{\mathbf{x}} = \frac{1}{n} \sum_{i=1}^n \delta_{x_i}$.
- 2 Extend $\tilde{V}_n$ to $\mathcal{V}_n : [0, T] \times \mathcal{P}_2(\mathbb{R}^d) \rightarrow \mathbb{R}$.
- 3 Show that $\mathcal{V}_n$ is equicontinuous and bounded in $[0, T] \times \mathcal{P}_r(\mathbb{R}^d)$, $r \in [1, 2)$.
- 4 $\mathcal{P}_2(\mathbb{R}^d) \subset \mathcal{P}_r(\mathbb{R}^d)$ compactly $\implies \mathcal{V}_n \rightarrow \mathcal{V}$ on bounded subsets of $\mathcal{P}_2(\mathbb{R}^d)$.
- 5 Define lift $V : [0, T] \times E \rightarrow \mathbb{R}$ by $V(t, X) := \mathcal{V}(t, X_{\#}\mathcal{L}^1)$
- 6 Show that V satisfies $(\text{HJB}_{\uparrow})$ and hence $V = U$.

Theorem (Convergence)

For every bounded set B in $\mathcal{P}_2(\mathbb{R}^d)$, we have

$$\lim_{n \rightarrow \infty} \sup \left\{ \left| u_n(t, x_1, \dots, x_n) - \mathcal{V} \left(t, \frac{1}{n} \sum_{i=1}^n \delta_{x_i} \right) \right| : (t, x_1, \dots, x_n) \in (0, T] \times (\mathbb{R}^d)^n, \frac{1}{n} \sum_{i=1}^n \delta_{x_i} \in B \right\} = 0$$

and $\mathcal{V}$ is the unique L -viscosity solution of (HJB_{∞}) . Moreover, $V = U$.

$C^{1,1}$ Regularity of U

Theorem (Lasry, Lions (1986))

$v : H \rightarrow \mathbb{R}$ *semiconcave, semiconvex, continuous* $\implies v \in C^{1,1}(H)$.

Proposition (Semiconcavity)

Let all coefficients be Lipschitz w.r.t. the $d_1(\cdot, \cdot)$ -distance and $C^{1,1}$. Then, for every t , $U(t, \cdot)$ is semiconcave with the semiconcavity constant independent of t .

Proposition (Semiconvexity: Case 1)

*Let all coefficients be Lipschitz w.r.t. the $d_1(\cdot, \cdot)$ -distance and $C^{1,1}$. Let the control cost be **strongly convex**. Then, for every t , $U(t, \cdot)$ is semiconvex with the semiconvexity constant independent of t .*

Proposition (Semiconvexity: Case 2)

*Let the state equation be **affine linear**, and the costs be $C^{1,1}$ and convex. Then, for every t , $U(t, \cdot)$ is convex.*

Projection

For $n \in \mathbb{N}$, $i \in \{1, \dots, n\}$, let $A_i^n = (\frac{i-1}{n}, \frac{i}{n}) \subset \Omega$. Now, we introduce $V_n : [0, T] \times (\mathbb{R}^d)^n \rightarrow \mathbb{R}$,

$$V_n(t, x_1, \dots, x_n) := V\left(t, \sum_{i=1}^n x_i \mathbf{1}_{A_i^n}\right).$$

Theorem (Projection under $C^{1,1}$ regularity)

Let $V(t, \cdot) \in C^{1,1}(E)$, for every t . Then, it holds

$$V_n(t, \mathbf{x}) = u_n(t, \mathbf{x}), \quad (t, \mathbf{x}) \in [0, T] \times (\mathbb{R}^d)^n.$$

Theorem (Projection without $C^{1,1}$ regularity)

Let the state equation be affine linear and let the cost be convex. Then, it holds

$$V_n(t, \mathbf{x}) = u_n(t, \mathbf{x}), \quad (t, \mathbf{x}) \in [0, T] \times (\mathbb{R}^d)^n.$$

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