Finite Dimensional Projections of HJB Equations in the Wasserstein Space

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SIAM Conference on Analysis of Partial Differential Equations
MS17 Mean Field Games and Control
Pittsburgh, PA, November 17-20, 2025

Georgia Institute of Technology

Finite Dimensional Control Problem

Fix $0 \le t < T < \infty$ and $\mathbf{x} = (x_1, \dots, x_n) \in (\mathbb{R}^d)^n$. Minimize

$$J_n(t, \mathbf{x}; \mathbf{a}(\cdot)) := \mathbb{E}\left[\int_t^T \left(\frac{1}{n} \sum_{i=1}^n \left(l_1(X_i(s), \mu_{\mathbf{X}(s)}) + l_2(a_i(s)) \right) \right) \mathrm{d}s + \mathcal{U}_T(\mu_{\mathbf{X}(T)}) \right]$$

subject to

$$\begin{cases} \mathrm{d}X_i(s) = [-a_i(s) + b(X_i(s), \mu_{\mathbf{X}(s)})]\mathrm{d}s + \sigma(X_i(s), \mu_{\mathbf{X}(s)})\mathrm{d}W(s) \\ X_i(t) = x_i \in \mathbb{R}^d, \end{cases}$$

$i = 1, \dots, n$. Here:

- $\mu_{\mathbf{x}} := \frac{1}{n} \sum_{i=1}^n \delta_{x_i}$ empirical measure
- $(W(s))_{s \in [t,T]}$ Wiener process – common noise.
- $\mathbf{a}(\cdot) = (a_1(\cdot), \dots, a_n(\cdot)) : [t, T] \times \Omega' \to (\mathbb{R}^d)^n$ control
- $l_1 : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ and $l_2 : \mathbb{R}^d \to \mathbb{R}$ running cost
- $\mathcal{U}_T : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ terminal cost

Dynamic Programming

Define **value function**

$$u_n(t,\mathbf{x}) := \inf_{\mathbf{a}(\cdot)} J_n(t,\mathbf{x};\mathbf{a}(\cdot)).$$

Satisfies **Hamilton–Jacobi–Bellman equation**

$$\begin{cases} \partial_t u_n + \frac{1}{2}\mathrm{Tr}(A_n(\mathbf{x},\mu_{\mathbf{x}})D^2 u_n) - \frac{1}{n}\sum_{i=1}^n H(x_i,\mu_{\mathbf{x}}, nD_{x_i}u_n) = 0 \\ u_n(T,\mathbf{x}) = \mathcal{U}_T(\mu_{\mathbf{x}}), \quad \mathbf{x}\in(\mathbb{R}^d)^n, \end{cases} \tag{HJB$_n$}$$

where $A_n(\mathbf{x},\mu)$ is $nd\times nd$-square matrix, Hamiltonian $H : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}$ is given by

$$H(x,\mu,p) = -b(x,\mu)\cdot p - l_1(x,\mu) + \sup_{q\in\mathbb{R}^d}(q\cdot p - l_2(q)).$$

Example: $n=2$, $d=1$, $\sigma\equiv 1$. Then

$$A_2(\mathbf{x},\mu_{\mathbf{x}}) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

$\rightsquigarrow$ Degenerate!

Limiting Equation

Formally u_n converges to solution of

$$\begin{cases} \partial_t \mathcal{U}(t,\mu) + \frac{1}{2}\int_{\mathbb{R}^d} \mathrm{Tr}\left[D_x \partial_\mu \mathcal{U}(t,\mu)(x)\sigma(x,\mu)\sigma^\top(x,\mu)\right]\mu(\mathrm{d}x) \\ \quad +\frac{1}{2}\int_{(\mathbb{R}^d)^2} \mathrm{Tr}\left[\partial_\mu^2 \mathcal{U}(t,\mu)(x,x')\sigma(x,\mu)\sigma^\top(x',\mu)\right]\mu(\mathrm{d}x)\mu(\mathrm{d}x') \\ \quad -\int_{\mathbb{R}^d} H(x,\mu,\partial_\mu \mathcal{U}(t,\mu)(x))\mu(\mathrm{d}x) = 0 \\ \mathcal{U}(T,\mu) = \mathcal{U}_T(\mu), \quad \mu \in \mathcal{P}_2(\mathbb{R}^d) \end{cases} \tag{HJB$_\infty$}$$

PDE on Wasserstein space $\mathcal{P}_2(\mathbb{R}^d) \rightsquigarrow$ hard to solve.

Lions' lift:

$$\mathcal{P}_2(\mathbb{R}^d) \qquad\qquad E := L^2(\Omega;\mathbb{R}^d), \quad \Omega = (0,1)$$

$$\mathcal{V} : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R} \qquad\qquad V : E \to \mathbb{R}, \quad V(X) := \mathcal{V}(X_\# \mathcal{L}^1)$$

Lifted equation:

$$\begin{cases} \partial_t V + \frac{1}{2}\mathrm{Tr}(\Sigma(X)(\Sigma(X))^* D^2 V) - \tilde{H}(X, X_\# \mathcal{L}^1, DV) = 0 \\ V(T,X) = U_T(X), \quad X \in E, \end{cases} \tag{HJB$_\uparrow$}$$

where U_T is lift of $\mathcal{U}_T$.

Infinite Dimensional Control Problem

Consider infinite dimensional SDE

$$\begin{cases} \mathrm{d}X(s) = [-a(s) + B(X(s))]\mathrm{d}s + \Sigma(X(s))\mathrm{d}W(s) \\ X(t) = X \in E, \end{cases}$$

where

$$B : E \to E \qquad B(X)(\omega) := b(X(\omega), X_{\#}\mathcal{L}^1)$$

$$\Sigma : E \to L(\mathbb{R}^{d'}, E) \qquad \Sigma(X)(\omega) := \sigma(X(\omega), X_{\#}\mathcal{L}^1)$$

Cost functional

$$J(t, X; a(\cdot)) = \mathbb{E}\left[\int_t^T \left(L_1(X(s)) + L_2(a(s))\right)\mathrm{d}s + U_T(X(T))\right],$$

where

$$L_1(X) := \int_\Omega l_1(X(\omega), X_{\#}\mathcal{L}^1)\mathrm{d}\omega, \quad L_2(X) := \int_\Omega l_2(X(\omega))\mathrm{d}\omega.$$

Define value function

$$U(t, X) := \inf_{a(\cdot)} J(t, X, a(\cdot)).$$

Main Results

Objects of interest:

- $u_n : [0,T] \times (\mathbb{R}^d)^n \to \mathbb{R}$ – finite-dim value functions
- $U : [0,T] \times L^2(\Omega;\mathbb{R}^d) \to \mathbb{R}$ – infinite-dim value function

How to connect these two?

- Define $\tilde{\mathcal{V}}_n(t,\mu_{\mathbf{x}}) := u_n(t,\mathbf{x}), \quad \mu_{\mathbf{x}} = \frac{1}{n}\sum_{i=1}^n \delta_{x_i}$.
- Extend $\tilde{\mathcal{V}}_n$ to a function $\mathcal{V}_n : [0,T] \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$

Main results:

1. Convergence of $\mathcal{V}_n \to \mathcal{V}$, where $\mathcal{V}$ lifts to U.
2. $C^{1,1}$ regularity of U.
3. Exact projection of U onto u_n.
4. Lifting and projection of optimal controls.

Convergence

1. Define $\tilde{\mathcal{V}}_n(t,\mu_{\mathbf{x}}) := u_n(t,\mathbf{x}), \quad \mu_{\mathbf{x}} = \frac{1}{n}\sum_{i=1}^n \delta_{x_i}$.
2. Extend $\tilde{\mathcal{V}}_n$ to $\mathcal{V}_n : [0,T] \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$.
3. Show that $\mathcal{V}_n$ is equicontinuous and bounded in $[0,T] \times \mathcal{P}_r(\mathbb{R}^d)$, $r \in [1,2)$.
4. $\mathcal{P}_2(\mathbb{R}^d) \subset \mathcal{P}_r(\mathbb{R}^d)$ compactly $\implies$ $\mathcal{V}_n \to \mathcal{V}$ on bounded subsets of $\mathcal{P}_2(\mathbb{R}^d)$.
5. Define lift $V : [0,T] \times E \to \mathbb{R}$ by $V(t,X) := \mathcal{V}(t, X_\# \mathcal{L}^1)$
6. Show that V satisfies (HJB$_\uparrow$) and hence $V = U$.

Theorem (Convergence)

For every bounded set B in $\mathcal{P}_2(\mathbb{R}^d)$, we have

$$\lim_{n\to\infty} \sup \left\{ \left| u_n(t,x_1,\dots,x_n) - \mathcal{V}\left(t, \frac{1}{n}\sum_{i=1}^n \delta_{x_i}\right) \right| \right.$$
$$\left. : (t,x_1,\dots,x_n) \in (0,T] \times (\mathbb{R}^d)^n, \frac{1}{n}\sum_{i=1}^n \delta_{x_i} \in B \right\} = 0$$

and $\mathcal{V}$ is the unique L-viscosity solution of (HJB$_\infty$). Moreover, $V = U$.

$C^{1,1}$ Regularity of U

Theorem (Lasry, Lions (1986))

$v : H \to \mathbb{R}$ *semiconcave, semiconvex, continuous* $\implies v \in C^{1,1}(H)$.

Proposition (Semiconcavity)

Let all coefficients be Lipschitz w.r.t. the $d_1(\cdot,\cdot)$-distance and $C^{1,1}$. Then, for every t, $U(t,\cdot)$ is semiconcave with the semiconcavity constant independent of t.

Proposition (Semiconvexity: Case 1)

Let all coefficients be Lipschitz w.r.t. the $d_1(\cdot,\cdot)$-distance and $C^{1,1}$. Let the control cost be ***strongly convex****. Then, for every t, $U(t,\cdot)$ is semiconvex with the semiconvexity constant independent of t.*

Proposition (Semiconvexity: Case 2)

Let the state equation be ***affine linear****, and the cost be $C^{1,1}$ and convex. Then, for every t, $U(t,\cdot)$ is convex.*

Projection

For $n \in \mathbb{N}$, $i \in \{1, \dots, n\}$, let $A_i^n = \left(\frac{i-1}{n}, \frac{i}{n}\right) \subset \Omega$. Now, we introduce $V_n : [0,T] \times (\mathbb{R}^d)^n \to \mathbb{R}$,

$$V_n(t, x_1, \dots, x_n) := V\left(t, \sum_{i=1}^n x_i \mathbf{1}_{A_i^n}\right).$$

Theorem (Projection under $C^{1,1}$ regularity)

Let $V(t,\cdot) \in C^{1,1}(E)$, for every t. Then, it holds

$$V_n(t, \mathbf{x}) = u_n(t, \mathbf{x}), \quad (t, \mathbf{x}) \in [0,T] \times (\mathbb{R}^d)^n.$$

Theorem (Projection without $C^{1,1}$ regularity)

Let the state equation be affine linear and let the cost be convex. Then, it holds

$$V_n(t, \mathbf{x}) = u_n(t, \mathbf{x}), \quad (t, \mathbf{x}) \in [0,T] \times (\mathbb{R}^d)^n.$$

Outlook: Mean-Field Control in Hilbert Spaces

Fix $0 \leq t < T < \infty$ and $\mathbf{x} = (x_1, \dots, x_n) \in H^n$. Minimize

$$J_n(t, \mathbf{x}; \mathbf{a}(\cdot)) = \mathbb{E}\left[\int_t^T \frac{1}{n}\sum_{i=1}^n l(x_i(s), \mu_{\mathbf{x}(s)}, a_i(s))\mathrm{d}s + \frac{1}{n}\sum_{i=1}^n \mathcal{U}_T(x_i(T), \mu_{\mathbf{x}(T)})\right]$$

subject to

$$\begin{cases} \mathrm{d}x_i(s) = [Ax_i(s) + f(x_i(s), \mu_{\mathbf{x}(s)}, a_i(s))]\mathrm{d}s + \sigma(x_i(s), \mu_{\mathbf{x}(s)})\mathrm{d}W(s), \quad s \in [t, T] \\ x_i(t) = x_i \in H, \end{cases}$$

$i = 1, \dots, n$. Here:

- H real, separable Hilbert space.
- $A : \mathcal{D}(A) \subset H \to H$ unbounded operator

Outlook: Mean-Field Control in Hilbert Spaces

Limiting equation on Wasserstein space:

$$\begin{cases} \partial_t \mathcal{U}(t,\mu) + \int_H \langle Ax, \partial_\mu \mathcal{U}(t,\mu)(x)\rangle \mu(\mathrm{d}x) \\ \quad + \frac{1}{2}\int_H \mathrm{Tr}\left[D_x \partial_\mu \mathcal{U}(t,\mu)(x)\sigma(x,\mu)\sigma^\top(x,\mu)\right]\mu(\mathrm{d}x) \\ \quad + \frac{1}{2}\int_{H\times H} \mathrm{Tr}\left[\partial_\mu^2 \mathcal{U}(t,\mu)(x,x')\sigma(x,\mu)\sigma^\top(x',\mu)\right]\mu(\mathrm{d}x)\mu(\mathrm{d}x') \\ \quad + \int_H \mathcal{H}(x,\mu,\partial_\mu \mathcal{U}(t,\mu)(x))\mu(\mathrm{d}x) = 0, \quad (t,\mu) \in (0,T)\times \mathcal{P}_2(H) \\ \mathcal{U}(T,\mu) = \int_H \mathcal{U}_T(x,\mu)\mu(\mathrm{d}x), \quad \mu \in \mathcal{P}_2(H). \end{cases}$$

Lifting procedure: Let $E = L^2(\Omega; H)$. Lifted HJB

$$\begin{cases} \partial_t V + \frac{1}{2}\mathrm{Tr}(\Sigma(X)(\Sigma(X))^* D^2 V) + \langle\!\langle \mathcal{A}X, DV \rangle\!\rangle + \tilde{\mathcal{H}}(X, DV) = 0 \\ V(T,X) = U_T(X), \quad X \in E, \end{cases}$$

where $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset E \to E$ is given by $\mathcal{A}(X)(\omega) := A(X(\omega))$.

New challenges:

- Loss of local compactness in H.
- Need to work with weaker space $H_{-1} \supset H$

References

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