Mean Field Games in Hilbert Spaces with Degenerate Diffusion: A Viscosity Solution Approach

Lukas Wessels

joint work with A. Święch

www.lukaswessels.org

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Symmetric N-Player Games

State equation: Players modeled by

$$\mathrm{d}X^i(s) = b(X^i(s), \alpha^i(s), \mu_{\mathbf{X}(s)})\mathrm{d}s + \sigma(X^i(s))\mathrm{d}W^i(s), \quad i = 1, \dots, N$$

where $\mu_{\mathbf{X}(s)} := \frac{1}{N}\sum_{j=1}^N \delta_{X^j(s)}$ and $\alpha^i(\cdot)$ denotes the control.

Cost Functional: Minimize

$$J^i(\alpha^1(\cdot), \dots, \alpha^N(\cdot)) = \mathbb{E}\left[\int_t^T f(X^i(s), \alpha^i(s), \mu_{\mathbf{X}(s)})\mathrm{d}s + G(X^i(T), \mu_{\mathbf{X}(T)})\right]$$

Nash Equilibrium: Controls $\alpha^{1,*}(\cdot), \dots, \alpha^{N,*}(\cdot)$ are Nash equilibrium, if

$$J^i(\alpha^{1,*}(\cdot), \dots, \alpha^{i,*}(\cdot), \dots, \alpha^{N,*}(\cdot)) \leq J^i(\alpha^{1,*}(\cdot), \dots, \alpha^{i}(\cdot), \dots, \alpha^{N,*}(\cdot))$$

for all controls $\alpha^i(\cdot)$, and for all $i \in \{1, \dots, N\}$.

Mean Field Games

State: Given a flow of measures $(m(s))_{s\in[t,T]}$, consider

$$\mathsf{d}X(s) = b(X(s),\alpha(s),m(s))\mathsf{d}s + \sigma(X(s))\mathsf{d}W(s), \quad X(t) = x.$$

where $\alpha(\cdot)$ denotes the control.

Value Function:

$$v(t,x) := \inf_{\alpha(\cdot)} \mathbb{E}\left[\int_t^T f(X(s),\alpha(s),m(s))\mathsf{d}s + G(X(T),m(T))\right]$$

Hamilton–Jacobi–Bellman Equation:

$$\begin{cases} \partial_t v(t,x) + \frac{1}{2}\mathsf{Tr}[\sigma(x)\sigma^*(x)D^2v(t,x)] - \mathcal{H}(x,Dv(t,x),m(t)) = 0 \\ v(T,\cdot) = G(\cdot,m(T)) \end{cases} \tag{HJB}$$

where the Hamiltonian is given by

$$\mathcal{H}(x,p,\mu) = \sup_{\alpha\in\Lambda_0}\{-\langle b(x,\alpha,\mu),p\rangle - f(x,\alpha,\mu)\}.$$

Mean Field Games – Continued

Hamiltonian:

$$\mathcal{H}(x,p,\mu) = \sup_{\alpha \in \Lambda_0} \{ -\langle b(x,\alpha,\mu), p \rangle - f(x,\alpha,\mu) \}. \tag{1}$$

Optimal control maximizes Hamiltonian:

$$\alpha^*(s,x) = \arg\max_{\alpha \in \Lambda_0} \{ -\langle b(x,\alpha,m(s)), Dv(s,x) \rangle - f(x,\alpha,m(s)) \}$$

Differentiating (1), we obtain optimal drift

$$b(x,\alpha^*(s,x),m(s)) = -\mathcal{H}_p(x,Dv(s,x),m(s)).$$

New state:

$$\mathsf{d}X(s) = -\mathcal{H}_p(x,Dv(s,x),m(s))\mathsf{d}s + \sigma(X(s))\mathsf{d}W(s), \quad X(t) = x.$$

Associated Fokker–Planck Equation:

$$\begin{cases} \partial_t m(t) - \frac{1}{2}\mathsf{Tr}[D^2(\sigma(x)\sigma^*(x)m(t))] - \mathsf{div}\,(\mathcal{H}_p(x,Dv(t,x),m(t))m(t)) = 0 \\ m(0) = m_0. \end{cases}$$

MFG System: Existence of Solutions

Nash equilibrium leads to coupled system of PDEs:

$$\partial_t v(t,x) + Lv(t,x) - \mathcal{H}(x, Dv(t,x), m(t)) = 0, \quad v(T,\cdot) = G(\cdot, m(T))$$
$$\partial_t m(t) - L^* m(t) - \operatorname{div}\left(\mathcal{H}_p(x, Dv(t,x), m(t)) m(t)\right) = 0, \quad m(0) = m_0 \in \mathcal{P}_1(H),$$

where

$$L\phi(x) = \frac{1}{2}\mathrm{Tr}[\sigma(x)\sigma^*(x)D^2\phi(x)].$$

How to construct solution?

1. Given $(m^n(s))_{s\in[t,T]}$, solve HJB equation to obtain v^n.
2. Given $\mathcal{H}_p(x, Dv^n(t,x), m^n(t))$, solve linear FP equation to obtain m^{n+1}.
3. Show that this procedure has a fixed point.

MFG System: Uniqueness of Solutions

Assumptions:

- Separated structure: $\mathcal{H}(x,p,\mu) = \mathcal{H}^0(x,p) - F(x,\mu)$.
- $\mathcal{H}^0$ is convex in p.
- F and G are Lasry–Lions monotone
- $\mathcal{H}^0$ strictly convex in p, i.e.,

$$\mathcal{H}^0(x,p_1) - \mathcal{H}^0(x,p_2) - \langle \mathcal{H}^0_p(x,p_2), p_1 - p_2 \rangle = 0 \quad \Longrightarrow \quad \mathcal{H}^0_p(x,p_1) = \mathcal{H}^0_p(x,p_2)$$

Classical argument: Let (v_1, m_1) and (v_2, m_2) be two solutions. Then $\bar{v} = v_1 - v_2$ satisfies

$$\partial_t \bar{v}(t,x) + L_0 \bar{v}(t,x) = \mathcal{H}(x, Dv_1(t,x), m_1(t)) - \mathcal{H}(x, Dv_2(t,x), m_2(t)).$$

MFG System: Uniqueness of Solutions – Continued

Using $\bar{v}$ as a test function, we obtain

$$\int_H \bar{v}(T,x)(m_1-m_2)(T,\mathrm{d}x) = \int_0^T \left(\int_H \left[\partial_t \bar{v}(t,x) + L_0 \bar{v}(t,x)\right] (m_1-m_2)(t,\mathrm{d}x) \right) \mathrm{d}t$$
$$- \int_0^T \left(\int_H \langle \mathcal{H}_p^0(x, Dv_1(s,x)),\ D\bar{v}(t,x)\rangle m_1(t,\mathrm{d}x) \right.$$
$$\left. - \int_H \langle \mathcal{H}_p^0(x, Dv_2(t,x)),\ D\bar{v}(t,x)\rangle m_2(t,\mathrm{d}x) \right) \mathrm{d}t$$

Now, we plug in the equation for $\bar{v}$, i.e.,

$$\partial_t \bar{v}(t,x) + L_0 \bar{v}(t,x) = \mathcal{H}(x, Dv_1(t,x), m_1(t)) - \mathcal{H}(x, Dv_2(t,x), m_2(t))$$

and use the separated structure of $\mathcal{H}$ to obtain...

MFG System: Uniqueness of Solutions – Continued 2

$$
\begin{aligned}
0 = &\int_H \left[G(x, m_1(T)) - G(x, m_2(T))\right](m_1 - m_2)(T, \mathrm{d}x) \\
&+ \int_0^T \int_H [F(x, m_1(t)) - F(x, m_2(t))](m_1 - m_2)(t, \mathrm{d}x)\mathrm{d}t \\
&+ \int_0^T \int_H \big[\mathcal{H}^0(x, Dv_2(t,x)) - \mathcal{H}^0(x, Dv_1(t,x)) \\
&\qquad\qquad - \langle \mathcal{H}^0_p(x, Dv_1(t,x)), -D\bar{v}(t,x)\rangle\big] m_1(t, \mathrm{d}x)\mathrm{d}t \\
&+ \int_0^T \int_H \big[\mathcal{H}^0(x, Dv_1(t,x)) - \mathcal{H}^0(x, Dv_2(t,x)) \\
&\qquad\qquad - \langle \mathcal{H}^0_p(x, Dv_2(t,x)),\ D\bar{v}(t,x)\rangle\big] m_2(t, \mathrm{d}x)\mathrm{d}t.
\end{aligned}
$$

Since G and F Lasry–Lions monotone, first two terms are non-negative. Since $\mathcal{H}^0$ convex in p, third and fourth term are non-negative. Due to strict convexity of $\mathcal{H}^0_p$, we obtain

$$\mathcal{H}^0_p(x, Dv_1(t,x)) = \mathcal{H}^0_p(x, Dv_2(t,x)) \quad m_1(t)- \text{ and } m_2(t)-\text{a.e. for } t \in [0,T].$$

$\implies m_1 = m_2$ by uniqueness of weak solutions of (5) with $w(t,x) = \mathcal{H}^0_p(x, Dv_1(t,x))$. Conclude that $v_1 = v_2$ using uniqueness of viscosity solutions of HJB equation.

Why Study MFG in Hilbert Spaces?

Consider

$$\begin{cases} \mathrm{d}y(s) = b_0\left(y(s), \int_{-d}^0 \eta(\theta)y(s+\theta)\mathrm{d}\theta, \alpha(s)\right)\mathrm{d}s + \sigma_0(y(s))\mathrm{d}W(s) \\ y(t) = x^0, \quad y(\theta) = x^1(\theta), \quad \theta \in [-d, 0) \end{cases}$$

Not Markovian! $\rightsquigarrow$ Introduce $H = \mathbb{R} \times L^2(-d,0)$; define $A : \mathcal{D}(A) \subset H \to H$,

$$Ax = \begin{bmatrix} 0 \\ x_1' \end{bmatrix}, \quad \mathcal{D}(A) = \left\{x = (x_0, x_1) \in H : x_1 \in W^{1,2}(-d,0), x_1(0) = x_0\right\}$$

and $b : H \times \Lambda_0 \to H$,

$$b(x,\alpha) = \begin{bmatrix} b_0\left(x_0, \int_{-d}^0 \eta(\theta)x_1(\theta)\mathrm{d}\theta, \alpha\right) \\ 0 \end{bmatrix}, \qquad \sigma(x) = \begin{bmatrix} \sigma_0(x_0) \\ 0 \end{bmatrix}.$$

Consider

$$\begin{cases} \mathrm{d}X(s) = [AX(s) + b(X(s), \alpha(s))]\mathrm{d}s + \sigma(X(s))\mathrm{d}W(s) \\ X(t) = x. \end{cases}$$

Why Study MFG in Hilbert Spaces? – Continued

Delay SDE:

$$\begin{cases} \mathrm{d}y(s) = b_0\left(y(s), \int_{-d}^{0} \eta(\theta) y(s+\theta)\mathrm{d}\theta, \alpha(s)\right)\mathrm{d}s + \sigma_0(y(s))\mathrm{d}W(s) \\ y(t) = x^0, \quad y(\theta) = x^1(\theta), \quad \theta \in [-d, 0) \end{cases} \tag{2}$$

Lifted SDE:

$$\begin{cases} \mathrm{d}X(s) = [AX(s) + b(X(s), \alpha(s))]\mathrm{d}s + \sigma(X(s))\mathrm{d}W(s) \\ X(t) = x. \end{cases} \tag{3}$$

Given controls α and initial condition $x = (x_0, x_1)$, let $y^{x,\alpha}(\cdot)$ be the solution of (2) and $X^{x,\alpha}(\cdot)$ be the solution of (3). Then, under appropriate assumptions, it holds

$$X^{x,\alpha}(t) = (y^{x,\alpha}(t), y^{x,\alpha}(t+\cdot)|_{[-d,0]}) \quad \forall t \geq 0.$$

MFG System in Hilbert Spaces

MFG system:

$$\begin{aligned} \partial_t v(t,x) + Lv(t,x) - \mathcal{H}(x, Dv(t,x), m(t)) = 0, &\quad v(T,\cdot) = G(\cdot, m(T)) \\ \partial_t m(t) - L^* m(t) - \operatorname{div}\left(\mathcal{H}_p(x, Dv(t,x), m(t)) m(t)\right) = 0, &\quad m(0) = m_0 \in \mathcal{P}_1(H) \end{aligned}$$

where

$$L\phi(x) := \langle Ax, D\phi(x) \rangle + \frac{1}{2}\operatorname{Tr}[\sigma(x)\sigma^*(x) D^2\phi(x)]$$

Difficulties:

- ⚠ Degenerate noise $\Longrightarrow$ No smooth solutions to HJB and FP (in general)
- ⚠ H is ∞-dim $\Longrightarrow$ Bounded sets are not precompact
- ⚠ A unbounded $\Longrightarrow$ Ax only defined for $x \in \mathcal{D}(A)$

MFG System in Hilbert Spaces: Notion of Solution

MFG system:

$$\partial_t v(t,x) + Lv(t,x) - \mathcal{H}(x, Dv(t,x), m(t)) = 0, \quad v(T,\cdot) = G(\cdot, m(T))$$
$$\partial_t m(t) - L^* m(t) - \operatorname{div}\left(\mathcal{H}_p(x, Dv(t,x), m(t))m(t)\right) = 0, \quad m(0) = m_0 \in \mathcal{P}_1(H)$$

Definition

A pair (v, m) is a solution of the MFG system (HJB)–(FP) if

- v *is viscosity solution of (HJB) satisfying* $v(t,\cdot) \in C_b^{1,1}(H_{-1})\ \forall t \in [0,T]$;
- m *is weak solution of linear FP equation*

$$\partial_t m(t) - L^* m(t) - \operatorname{div}(w(t,x)m(t)) = 0, \quad m(0) = m_0.$$

with $w(t,x) = \mathcal{H}_p(x, Dv(t,x), m(t))$.

Viscosity Solutions – Motivation

Consider one-dimensional HJB equation

$$\begin{cases} v_t(t,x) + \inf_\alpha \left\{ f(x,\alpha) + v_x(t,x) b(x,\alpha) + \frac{1}{2} v_{xx}(t,x)\sigma^2(x,\alpha) \right\} = 0 \\ v(T,x) = g(x), \quad x \in \mathbb{R}. \end{cases}$$

Let $\varphi \in C^\infty((0,T)\times\mathbb{R})$ be such that $v-\varphi$ has global maximum at (t,x), i.e.,

$$v(s,y) - v(t,x) \le \varphi(s,y) - \varphi(t,x)$$

for all $(s,y) \in [0,T]\times\mathbb{R}$. Then,

$$\begin{aligned} 0 &= \inf_{\alpha(\cdot)} \mathbb{E}\left[\int_t^\tau f(X(s),\alpha(s))\mathrm{d}s + v(\tau, X(\tau)) - v(t,x) \right] \\ &\le \inf_{\alpha(\cdot)} \mathbb{E}\left[\int_t^\tau f(X(s),\alpha(s))\mathrm{d}s + \varphi(\tau, X(\tau)) - \varphi(t,x) \right]. \end{aligned}$$

Thus,

$$\varphi_t(t,x) + \inf_\alpha \left\{ f(x,\alpha) + \varphi_x(t,x) b(x,\alpha) + \frac{1}{2}\varphi_{xx}(t,x)\sigma^2(x,\alpha) \right\} \ge 0.$$

Viscosity Solutions – Definition

Consider the HJB equation

$$\begin{cases} v_t(t,x) + \inf_\alpha \left\{ f(x,\alpha) + v_x(t,x)b(x,\alpha) + \frac{1}{2} v_{xx}(t,x)\sigma^2(x,\alpha) \right\} = 0 \\ v(T,x) = g(x). \end{cases} \tag{4}$$

Definition (Viscosity Solution)

$v \in C([0,T] \times \mathbb{R})$ *is a viscosity subsolution of* (4)*, if* $v(T,y) \leq g(y)$ *and* $\forall \varphi \in C^\infty((0,T) \times \mathbb{R})$ *such that* $v - \varphi$ *has a global maximum at* (t,x)*, it holds*

$$\varphi_t(t,x) + \inf_\alpha \left\{ f(x,\alpha) + \varphi_x(t,x)b(x,\alpha) + \frac{1}{2}\varphi_{xx}(t,x)\sigma^2(x,\alpha) \right\} \geq 0$$

Viscosity Solutions – Consistency

Let $v \in C^{1,2}((0,T) \times \mathbb{R})$ be a classical solution of

$$\begin{cases} v_t(t,x) + \inf_\alpha \left\{ f(x,\alpha) + v_x(t,x)b(x,\alpha) + \frac{1}{2} v_{xx}(t,x)\sigma^2(x,\alpha) \right\} = 0 \\ v(T,x) = g(x) \end{cases}$$

and let $\varphi \in C^\infty((0,T) \times \mathbb{R})$ be such that $v - \varphi$ has a global maximum at (t,x). Then

$$v_t(t,x) = \varphi_t(t,x), \quad v_x(t,x) = \varphi_x(t,x), \quad v_{xx}(t,x) \leq \varphi_{xx}(t,x).$$

Hence,

$$\begin{aligned} &\varphi_t(t,x) + \inf_\alpha \left\{ f(x,\alpha) + \varphi_x(t,x)b(x,\alpha) + \frac{1}{2}\varphi_{xx}(t,x)\sigma^2(x,\alpha) \right\} \\ &\geq v_t(t,x) + \inf_\alpha \left\{ f(x,\alpha) + v_x(t,x)b(x,\alpha) + \frac{1}{2}v_{xx}(t,x)\sigma^2(x,\alpha) \right\} = 0 \end{aligned}$$

$\Longrightarrow$ Every classical solution is a viscosity solution.

Excursion: B-Continuous Viscosity Solutions

Consider the PDE

$$\partial_t v + \langle Ax, Dv \rangle + \mathcal{F}\left(x, Dv, D^2 v\right) = 0, \quad v(T,x) = g(x)$$

where $A : \mathcal{D}(A) \subset H \to H$ is linear, densely defined, maximal dissipative.
Problem:

$$\langle Ax, D\varphi \rangle$$

is not well defined for generic test function $\varphi \in C^{1,2}([0,T] \times H)$!
Let $B \in L(H)$ be strictly positive, self-adjoint such that $A^* B \in L(H)$ and

$$-A^* B + c_0 B \geq 0, \quad \text{for some } c_0 \geq 0.$$

Unbounded A: Test with φ such that $D\varphi \in \mathcal{D}(A^*)$ and rewrite

$$\langle Ax, D\varphi \rangle = \langle x, A^* D\varphi \rangle.$$

Compactness: We assume that

$$\boxed{B \text{ is compact}}$$

and work in the space $H_{-1} =$ completion of H w.r.t. $\|x\|_{-1}^2 := \langle Bx, x \rangle$.

Weak Solutions for Fokker–Planck Equations

Linear FP equation:

$$\partial_t m(t) - L^* m(t) - \operatorname{div}\left(w(t,x)m(t)\right) = 0, \quad m(0) = m_0. \tag{5}$$

Definition 1

A flow of measures $(m(s))_{s\in[0,T]}$ is a weak solution of (5) with initial condition $m_0 \in \mathcal{P}_1(H)$ if

$$\begin{aligned}&\int_H \varphi(t,x)m(t,\mathrm{d}x) - \int_H \varphi(0,x)m_0(\mathrm{d}x)\\ &= \int_0^t \left(\int_H \left[\partial_t \varphi(s,x) + L\varphi(s,x) - \langle w(s,x), D\varphi(s,x)\rangle\right] m(s,\mathrm{d}x)\right) \mathrm{d}s\end{aligned}$$

for every $t \in [0,T]$ and for all test functions φ.

Uniqueness of Weak Solutions (Outline)

Let m_1 and m_2 be two solutions of the linear FP equation. Then

$$\begin{aligned}&\int_H \varphi(T,x)(m_1 - m_2)(T, \mathrm{d}x) \\ &= \int_0^T \left(\int_H \left[\partial_t \varphi(s,x) + L\varphi(s,x) - \langle w(s,x), D\varphi(s,x) \rangle \right] (m_1 - m_2)(s, \mathrm{d}x) \right) \mathrm{d}s\end{aligned}$$

Consider the adjoint equation

$$\partial_t u + Lu - \langle w(t,x), Du \rangle = f(t,x), \quad u(T, \cdot) = 0.$$

Assume that we can solve this equation for any right-hand side f and the solution is smooth. Then, we can test with $\varphi = u$ and obtain

$$0 = \int_0^T \left(\int_H f(s,x)(m_1 - m_2)(s, \mathrm{d}x) \right) \mathrm{d}s$$

for all f, hence $m_1 = m_2$.

Weak Solutions (Infinite Dimensional Case)

Class of test functions:

$$\mathcal{D}_T = \Big\{ \varphi \in C_b^{1,2}([0,T] \times H) : A^* D\varphi(t,x) \in C_b([0,T] \times H; H) \Big\}.$$

Adjoint equation:

$$\partial_t u + \langle Ax, Du \rangle + \frac{1}{2} \mathrm{Tr}[\sigma(x)\sigma^*(x) D^2 u] - \langle w(t,x), Du \rangle = f(t,x), \quad u(T,\cdot) = 0.$$

Problem: We cannot solve this equation such that $u \in \mathcal{D}_T$.
Workaround: Use approximation:

1. Approximate w and f so that we can obtain viscosity solutions u
2. Show that u is $C^{1,1}$.
3. Approximate u by $u_n \in \mathcal{D}_T$
4. Perform classical argument with u_n and pass to the limit.

$C^{1,1}$ Regularity

Theorem (Lasry, Lions (1986))

$v : H \to \mathbb{R}$ *semiconcave, semiconvex, continuous* $\implies v \in C^{1,1}(H)$.

Adjoint equation:

$$\partial_t u + \langle Ax, Du \rangle + \frac{1}{2}\mathrm{Tr}[\sigma(x)\sigma^*(x)D^2 u] - \langle w(t,x), Du \rangle = f(t,x), \quad u(T,\cdot) = 0.$$

Feynman–Kac representation:

$$u(t,x) = \mathbb{E}\left[\int_t^T f(s, X^{t,x}(s))\mathrm{d}s\right]$$

where

$$\mathrm{d}X(s) = [AX(s) - w(s, X(s))]\mathrm{d}s + \sigma(X(s))\mathrm{d}W(s), \quad X_t = x.$$

Now, semiconcavity/convexity can be transferred from f to u.

MFG System: Well-Posedness

Definition 2

A pair (v, m) is a solution of the MFG system (HJB)–(FP) if

- v is viscosity solution of (HJB) satisfying $v(t, \cdot) \in C_b^{1,1}(H_{-1})\ \forall t \in [0, T]$;
- m is solution of linear FP equation with $w(t, x) = \mathcal{H}_p(x, Dv(t, x), m(t))$.

Existence: Define

$$Q_{m_0} := \left\{ \mu \in \mathcal{P}_1(H) : \int_H \|x\|^2 \mu(\mathrm{d}x) \leq c_{m_0} \right\},$$

Then Q_{m_0} is compact in $\mathcal{P}_1(H_{-1})$. Now, define

$$C_{m_0} := \left\{ m : [0, T] \to Q_{m_0} : m(0) = m_0 \text{ and } \mathbf{d}_{1,-1}(m(t), m(s)) \leq M_1 \sqrt{|t - s|} \right\},$$

Then, C_{m_0} is convex and compact in $C([0, T]; \mathcal{M}(H_{-1}))$.
In order to prove existence, we apply Tikhonov's fixed point theorem.
Uniqueness: Approximate solutions by regular solutions and follow classical argument.

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