Mean Field Games in Hilbert Spaces with Degenerate Diffusion: A Viscosity Solution Approach

Lukas Wessels

joint work with A. Święch

www.lukaswessels.org

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Symmetric N-Player Games

State equation: Players $i = 1, \dots, N$ modeled by

$$\mathrm{d}X^i(s) = b(X^i(s), \alpha^i(s), \mu_{\mathbf{X}(s)})\mathrm{d}s + \sigma(X^i(s))\mathrm{d}W^i(s), \quad X^i(t) = x^i \quad \text{i.i.d.},$$

where $\mu_{\mathbf{X}(s)} := \frac{1}{N}\sum_{i=1}^N \delta_{X^i(s)}$ and $\alpha^i(\cdot)$ denotes the control.

Cost Functional: Minimize

$$J^i(\alpha^1(\cdot), \dots, \alpha^N(\cdot)) = \mathbb{E}\left[\int_t^T f(X^i(s), \alpha^i(s), \mu_{\mathbf{X}(s)})\mathrm{d}s + G(X^i(T), \mu_{\mathbf{X}(T)})\right]$$

Nash Equilibrium: Controls $\alpha^{1,*}(\cdot), \dots, \alpha^{N,*}(\cdot)$ are Nash equilibrium, if

$$J^i(\alpha^{1,*}(\cdot), \dots, \alpha^{i,*}(\cdot), \dots, \alpha^{N,*}(\cdot)) \leq J^i(\alpha^{1,*}(\cdot), \dots, \alpha^{i}(\cdot), \dots, \alpha^{N,*}(\cdot))$$

for all controls $\alpha^i(\cdot)$, and for all $i \in \{1, \dots, N\}$.

Mean Field Games

State equation: Given flow of measures $(m(s))_{s\in[t,T]}$, consider

$$\mathrm{d}X(s) = b(X(s),\alpha(s),m(s))\mathrm{d}s + \sigma(X(s))\mathrm{d}W(s), \quad X(t) = x.$$

Value function:

$$v(t,x) := \inf_{\alpha(\cdot)} \mathbb{E}\left[\int_t^T f(X(s),\alpha(s),m(s))\mathrm{d}s + G(X(T),m(T))\right]$$

MFG System: Search for an equilibrium leads to coupled system

$$\begin{cases} \partial_t v(t,x) + \frac{1}{2}\mathrm{Tr}[\sigma(x)\sigma^*(x)D^2 v(t,x)] - \mathcal{H}(x,Dv(t,x),m(t)) = 0 \\ \partial_t m(t) - \frac{1}{2}\mathrm{Tr}[D^2(\sigma(x)\sigma^*(x)m(t))] - \mathrm{div}\,(\mathcal{H}_p(x,Dv(t,x),m(t))m(t)) = 0 \\ v(T,\cdot) = G(\cdot,m(T)), \quad m(0) = m_0 \end{cases}$$

where the Hamiltonian is given by

$$\mathcal{H}(x,p,\mu) = \sup_{\alpha\in\Lambda_0}\{-\langle b(x,\alpha,\mu),p\rangle - f(x,\alpha,\mu)\}.$$

Why Study MFGs in Hilbert Spaces?

Example: Consider Stochastic Delay Differential Equation

$$\begin{cases} \mathrm{d}y(s) = b_0\left(y(s), \int_{-d}^0 \eta(\theta)y(s+\theta)\mathrm{d}\theta, \alpha(s)\right)\mathrm{d}s + \sigma_0(y(s))\mathrm{d}W(s) \\ y(t) = x^0, \quad y(\theta) = x^1(\theta), \quad \theta \in [-d, 0). \end{cases}$$

Not Markovian! To regain Markovianity, introduce

$$H = \underbrace{\mathbb{R}}_{\text{Present}} \times \underbrace{L^2(-d,0)}_{\text{Past}}$$

and define $A : \mathcal{D}(A) \subset H \to H$,

$$Ax = \begin{bmatrix} 0 \\ x_1' \end{bmatrix}, \quad \mathcal{D}(A) = \left\{x = (x_0, x_1) \in H : \ \ x_1 \in W^{1,2}(-d,0), \ \ x_1(0) = x_0\right\}$$

Consider

$$\mathrm{d}X(s) = [AX(s) + b(X(s), \alpha(s))]\mathrm{d}s + \sigma(X(s))\mathrm{d}W(s), \quad X(t) = x.$$

Then, under appropriate assumptions, it holds

$$X(s) = (y(s), y(s+\cdot)|_{[-d,0]}) \quad \forall s \geq t.$$

MFG System in Hilbert Spaces

MFG system:

$$\begin{cases} \partial_t v(t,x) + Lv(t,x) - \mathcal{H}(x, Dv(t,x), m(t)) = 0 \\ \partial_t m(t) - L^* m(t) - \mathsf{div}\left(\mathcal{H}_p(x, Dv(t,x), m(t)) m(t)\right) = 0 \\ v(T,\cdot) = G(\cdot, m(T)), \quad m(0) = m_0 \in \mathcal{P}_1(H) \end{cases}$$

where

$$L\phi(x) := \langle Ax, D\phi(x) \rangle + \frac{1}{2}\mathsf{Tr}[\sigma(x)\sigma^*(x) D^2\phi(x)]$$

Difficulties:

⚠	Degenerate noise	$\Longrightarrow$	No regular solutions to HJB and FP
⚠	H is ∞-dim	$\Longrightarrow$	No local compactness, no Lebesgue measure
⚠	A unbounded	$\Longrightarrow$	Ax only defined for $x \in \mathcal{D}(A)$

MFG System in Hilbert Spaces: Notion of Solution

MFG system:

$$\begin{cases} \partial_t v(t,x) + Lv(t,x) - \mathcal{H}(x, Dv(t,x), m(t)) = 0 \\ \partial_t m(t) - L^* m(t) - \mathrm{div}\,(\mathcal{H}_p(x, Dv(t,x), m(t)) m(t)) = 0 \\ v(T,\cdot) = G(\cdot, m(T)), \quad m(0) = m_0 \in \mathcal{P}_1(H) \end{cases} \qquad (*)$$

Definition

A pair (v, m) is a solution of the MFG system (HJB)–(FP) if

- *v is viscosity solution of HJB satisfying $v(t,\cdot) \in C_b^{1,1}(H_{-1})$ $\forall t \in [0,T]$;*
- *m is weak solution of linear FP equation*

$$\partial_t m(t) - L^* m(t) - \mathit{div}\,(w(t,x) m(t)) = 0, \quad m(0) = m_0.$$

with $w(t,x) = \mathcal{H}_p(x, Dv(t,x), m(t))$.

Main Result: Existence and Uniqueness of solution to MFG system $(*)$.

B-Continuous Viscosity Solutions

Consider the PDE

$$\partial_t v + \langle Ax, Dv \rangle + \frac{1}{2}\mathsf{Tr}[\sigma(x)\sigma^*(x)D^2 v] - \mathcal{H}(x, Dv, m(t)) = 0$$

Problem: $\langle Ax, D\varphi \rangle$ not well-defined.
Let $B \in L(H)$ be strictly positive, self-adjoint such that $A^* B \in L(H)$ and

$$-A^* B + c_0 B \geq 0, \quad \text{for some } c_0 \geq 0.$$

Work in the space $H_{-1} =$ completion of H w.r.t. $\|x\|_{-1}^2 := \langle Bx, x \rangle$.

Unbounded A: Test with φ such that $D\varphi \in \mathcal{D}(A^*)$ and rewrite

$$\langle Ax, D\varphi \rangle = \langle x, A^* D\varphi \rangle.$$

Compactness: We assume that

$$\boxed{B \text{ is compact}}$$

$C^{1,1}$ Regularity

Theorem (Lasry, Lions (1986))

$v : H \to \mathbb{R}$ semiconcave, semiconvex, continuous $\implies$ $v \in C^{1,1}(H)$.

Value function:

$$v(t,x) := \inf_{\alpha(\cdot)\in\mathcal{A}_t} \mathbb{E}\left[\int_t^T f(X(s),\alpha(s),m(s))\mathrm{d}s + G(X(T),m(T))\right]$$

subject to

$$\mathrm{d}X(s) = [AX(s) + b(X(s),\alpha(s),m(s))]\mathrm{d}s + \sigma(X(s))\mathrm{d}W(s), \quad X(t) = x \in H.$$

Theorem (de Feo, Święch, Wessels (2025))

- *v is semiconcave under suitable assumptions on the coefficients.*
- *v is semiconvex in each of the following two cases:*
  1. *Affine linear state equation + convex cost.*
  2. *$f(x,\alpha,\mu) = f_0(x,\mu) + \nu|\alpha|^2$, ν large (or T small)*

Weak Solutions for Fokker–Planck Equations

Linear FP equation:

$$\partial_t m(t) - L^* m(t) - \mathrm{div}\,(w(t,x)m(t)) = 0, \quad m(0) = m_0. \tag{1}$$

Definition

A flow of measures $(m(s))_{s\in[0,T]}$ *is a weak solution of* (1) *with initial condition* $m_0 \in \mathcal{P}_1(H)$ *if*

$$\begin{aligned}&\int_H \varphi(t,x)m(t,\mathrm{d}x) - \int_H \varphi(0,x)m_0(\mathrm{d}x)\\ &= \int_0^t \left(\int_H \left[\partial_t \varphi(s,x) + L\varphi(s,x) - \langle w(s,x), D\varphi(s,x)\rangle\right] m(s,\mathrm{d}x) \right) \mathrm{d}s\end{aligned}$$

for every $t \in [0,T]$ *and for all test functions* φ.

Uniqueness of Weak Solutions – Outline

Let m_1 and m_2 be two solutions of the linear FP equation. Then

$$\begin{aligned}&\int_H \varphi(T,x)(m_1 - m_2)(T,\mathrm{d}x) \\ &= \int_0^T \left(\int_H \left[\partial_t \varphi(s,x) + L\varphi(s,x) - \langle w(s,x), D\varphi(s,x)\rangle\right](m_1 - m_2)(s,\mathrm{d}x)\right)\mathrm{d}s.\end{aligned} \tag{2}$$

Consider the adjoint equation

$$\partial_t u + Lu - \langle w(t,x), Du\rangle = f(t,x), \quad u(T,\cdot) = 0.$$

Assume we can solve this equation for any f, and u is smooth.

Testing (2) with $\varphi = u$ yields

$$0 = \int_0^T \left(\int_H f(s,x)(m_1 - m_2)(s,\mathrm{d}x)\right)\mathrm{d}s$$

for all f, hence $m_1 = m_2$.

Weak Solutions – Infinite Dimensional Case

Class of test functions:

$$\mathcal{D}_T = \Big\{\varphi \in C_b^{1,2}([0,T] \times H) : A^* D\varphi(t,x) \in C_b([0,T] \times H; H)\Big\}.$$

Adjoint equation:

$$\partial_t u + \langle Ax, Du \rangle + \frac{1}{2}\mathrm{Tr}[\sigma(x)\sigma^*(x)D^2 u] - \langle w(t,x), Du \rangle = f(t,x), \quad u(T,\cdot) = 0.$$

Problem: We cannot solve this equation such that $u \in \mathcal{D}_T$.
Workaround: Use approximation:

1. Approximate w and f so that we can obtain viscosity solution u that is $C^{1,1}$.
2. Approximate u by $u_n \in \mathcal{D}_T$
3. Perform classical argument with u_n and pass to the limit.

MFG System: Well-Posedness

Definition

A pair (v, m) *is a solution of the MFG system (HJB)–(FP) if*

- v *is viscosity solution of (HJB) satisfying* $v(t, \cdot) \in C_b^{1,1}(H_{-1})\ \forall t \in [0, T]$*;*
- m *is solution of linear FP equation with* $w(t, x) = \mathcal{H}_p(x, Dv(t, x), m(t))$*.*

Existence: Define mapping Ψ on a subset of $C([0, T]; \mathcal{P}_1(H_{-1}))$ as follows:

$$m \ \rightsquigarrow\ v^{(m)} \text{ solution of HJB } \ \rightsquigarrow\ \Psi(m) \text{ solution of FP.}$$

To apply Tikhonov's fixed point theorem, we need:

- $\Psi : C \to C$ for some $C \subset C([0, T]; \mathcal{M}(H_{-1}))$ convex and compact;
- Ψ continuous.

Uniqueness: Approximate solutions by regular solutions and follow classical argument based on Lasry–Lions type monotonicity.

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