

Mean Field Games in Hilbert Spaces with Degenerate Diffusion: A Viscosity Solution Approach

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Mean Field Games in Hilbert Spaces

State equation: Given flow of measures $(m(s))_{s \in [t, T]}$, consider

$$dX(s) = [AX(s) + b(X(s), \alpha(s), m(s))]ds + \sigma(X(s))dW(s), \quad X(t) = x \in H.$$

Value function: $v : [0, T] \times H \rightarrow \mathbb{R}$ defined as

$$v(t, x) := \inf_{\alpha(\cdot)} \mathbb{E} \left[\int_t^T f(X(s), \alpha(s), m(s))ds + G(X(T), m(T)) \right].$$

MFG System: Search for Nash equilibria leads to coupled system

$$\begin{cases} \partial_t v(t, x) + Lv(t, x) - \mathcal{H}(x, Dv(t, x), m(t)) = 0 \\ \partial_t m(t) - L^*m(t) - \operatorname{div}(\mathcal{H}_p(x, Dv(t, x), m(t))m(t)) = 0 \\ v(T, \cdot) = G(\cdot, m(T)), \quad m(0) = m_0 \in \mathcal{P}_1(H) \end{cases}$$

where

- $L\phi(x) := \langle Ax, D\phi(x) \rangle + \frac{1}{2} \operatorname{Tr}[\sigma(x)\sigma^*(x)D^2\phi(x)]$
- $\mathcal{H}(x, p, \mu) = \sup_{\alpha \in \Lambda_0} \{ -\langle b(x, \alpha, \mu), p \rangle - f(x, \alpha, \mu) \}$

Why Study MFGs in Hilbert Spaces?

Example: Consider Stochastic Delay Differential Equation

$$\begin{cases} dy(s) = b_0 \left(y(s), \int_{-d}^0 \eta(\theta) y(s + \theta) d\theta, \alpha(s) \right) ds + \sigma_0(y(s)) dW(s) \\ y(t) = x^0, \quad y(\theta) = x^1(\theta), \quad \theta \in [-d, 0). \end{cases}$$

Not Markovian! To regain Markovianity, introduce

$$H = \underbrace{\mathbb{R}}_{\text{Present}} \times \underbrace{L^2(-d, 0)}_{\text{Past}}$$

and define $A : \mathcal{D}(A) \subset H \rightarrow H$,

$$Ax = \begin{bmatrix} 0 \\ x'_1 \end{bmatrix}, \quad \mathcal{D}(A) = \{x = (x_0, x_1) \in H : x_1 \in W^{1,2}(-d, 0), x_1(0) = x_0\}$$

Consider

$$dX(s) = [AX(s) + b(X(s), \alpha(s))]ds + \sigma(X(s))dW(s), \quad X(t) = x.$$

Then, under appropriate assumptions, it holds

$$X(s) = (y(s), y(s + \cdot)|_{[-d, 0]}) \quad \forall s \geq t.$$

MFGs in Hilbert Spaces: Challenges

MFG system:

$$\begin{cases} \partial_t v(t, x) + Lv(t, x) - \mathcal{H}(x, Dv(t, x), m(t)) = 0 \\ \partial_t m(t) - L^* m(t) - \operatorname{div}(\mathcal{H}_p(x, Dv(t, x), m(t))m(t)) = 0 \\ v(T, \cdot) = G(\cdot, m(T)), \quad m(0) = m_0 \in \mathcal{P}_1(H) \end{cases}$$

where

$$L\phi(x) := \langle Ax, D\phi(x) \rangle + \frac{1}{2} \operatorname{Tr}[\sigma(x)\sigma^*(x)D^2\phi(x)]$$

Challenges:

- ⚠ H is ∞ -dim $\implies$ No local compactness, no Lebesgue measure
- ⚠ A unbounded $\implies Ax$ only defined for $x \in \mathcal{D}(A)$
- ⚠ Degenerate noise $\implies$ No regular solutions to HJB and FP

MFG System: Notion of Solution

MFG system:

$$\begin{cases} \partial_t v(t, x) + Lv(t, x) - \mathcal{H}(x, Dv(t, x), m(t)) = 0 \\ \partial_t m(t) - L^*m(t) - \operatorname{div}(\mathcal{H}_p(x, Dv(t, x), m(t))m(t)) = 0 \\ v(T, \cdot) = G(\cdot, m(T)), \quad m(0) = m_0 \in \mathcal{P}_1(H) \end{cases} \quad (*)$$

Definition

A pair (v, m) is a solution of the MFG system $(*)$ if

- v is viscosity solution of HJB satisfying $v(t, \cdot) \in C_b^{1,1}(H_{-1}) \ \forall t \in [0, T]$;
- m is weak solution of linear FP equation

$$\partial_t m(t) - L^*m(t) - \operatorname{div}(w(t, x)m(t)) = 0, \quad m(0) = m_0.$$

with $w(t, x) = \mathcal{H}_p(x, Dv(t, x), m(t))$.

Main Result: Existence and Uniqueness of solution to MFG system .

B -Continuous Viscosity Solutions

Consider the PDE

$$\partial_t v + \langle Ax, Dv \rangle + \frac{1}{2} \text{Tr}[\sigma(x)\sigma^*(x)D^2v] - \mathcal{H}(x, Dv, m(t)) = 0$$

Problem: $\langle Ax, D\varphi \rangle$ not well-defined.

Let $B \in L(H)$ be strictly positive, self-adjoint such that $A^*B \in L(H)$ and

$$-A^*B + c_0B \geq 0, \quad \text{for some } c_0 \geq 0.$$

Work in the space H_{-1} = completion of H w.r.t. $\|x\|_{-1}^2 := \langle Bx, x \rangle$.

Unbounded A : Test with φ such that $D\varphi \in \mathcal{D}(A^*)$ and rewrite

$$\langle Ax, D\varphi \rangle = \langle x, A^*D\varphi \rangle.$$

Compactness: We assume that

B is compact

Theorem (Lasry, Lions (1986))

$v : H \rightarrow \mathbb{R}$ *semiconcave, semiconvex, continuous* $\implies v \in C^{1,1}(H)$.

Value function:

$$v(t, x) := \inf_{\alpha(\cdot) \in \mathcal{A}_t} \mathbb{E} \left[\int_t^T f(X(s), \alpha(s), m(s)) ds + G(X(T), m(T)) \right]$$

subject to

$$dX(s) = [AX(s) + b(X(s), \alpha(s), m(s))]ds + \sigma(X(s))dW(s), \quad X(t) = x \in H.$$

Theorem (de Feo, Świąch, Wessels (2025))

- v is semiconcave under suitable assumptions on the coefficients.
- v is semiconvex in each of the following two cases:
 - 1 Affine linear state equation + convex cost.
 - 2 $f(x, \alpha, \mu) = f_0(x, \mu) + \nu|\alpha|^2$, ν large (or T small)

Weak Solutions for Fokker–Planck Equations

Linear FP equation:

$$\partial_t m(t) - L^* m(t) - \operatorname{div}(w(t, x)m(t)) = 0, \quad m(0) = m_0. \quad (1)$$

Definition

A flow of measures $(m(s))_{s \in [0, T]}$ is a weak solution of (1) with initial condition $m_0 \in \mathcal{P}_1(H)$ if

$$\begin{aligned} & \int_H \varphi(t, x) m(t, dx) - \int_H \varphi(0, x) m_0(dx) \\ &= \int_0^t \left(\int_H [\partial_t \varphi(s, x) + L\varphi(s, x) - \langle w(s, x), D\varphi(s, x) \rangle] m(s, dx) \right) ds \end{aligned}$$

for every $t \in [0, T]$ and for all test functions φ .

Uniqueness of Weak Solutions – Outline

Let m_1 and m_2 be two solutions of the linear FP equation. Then

$$\begin{aligned} & \int_H \varphi(T, x)(m_1 - m_2)(T, dx) \\ &= \int_0^T \left(\int_H [\partial_t \varphi(s, x) + L\varphi(s, x) - \langle w(s, x), D\varphi(s, x) \rangle] (m_1 - m_2)(s, dx) \right) ds. \end{aligned} \tag{2}$$

Consider the adjoint equation

$$\partial_t u + Lu - \langle w(t, x), Du \rangle = f(t, x), \quad u(T, \cdot) = 0.$$

Assume we can solve this equation for any f , and u is smooth.

Testing (2) with $\varphi = u$ yields

$$0 = \int_0^T \left(\int_H f(s, x)(m_1 - m_2)(s, dx) \right) ds$$

for all f , hence $m_1 = m_2$.

Weak Solutions – Infinite Dimensional Case

Class of test functions:

$$\mathcal{D}_T = \left\{ \varphi \in C_b^{1,2}([0, T] \times H) : A^* D\varphi(t, x) \in C_b([0, T] \times H; H) \right\}.$$

Adjoint equation:

$$\partial_t u + \langle Ax, Du \rangle + \frac{1}{2} \text{Tr}[\sigma(x)\sigma^*(x)D^2u] - \langle w(t, x), Du \rangle = f(t, x), \quad u(T, \cdot) = 0.$$

Problem: We cannot solve this equation such that $u \in \mathcal{D}_T$.

Workaround: Use approximation:

- ① Approximate w and f so that we can obtain viscosity solution u that is $C^{1,1}$.
- ② Approximate u by $u_n \in \mathcal{D}_T$
- ③ Perform classical argument with u_n and pass to the limit.

Definition

A pair (v, m) is a solution of the MFG system (HJB)–(FP) if

- v is viscosity solution of (HJB) satisfying $v(t, \cdot) \in C_b^{1,1}(H_{-1}) \forall t \in [0, T]$;
- m is solution of linear FP equation with $w(t, x) = \mathcal{H}_p(x, Dv(t, x), m(t))$.

Existence: Define mapping Ψ on a subset of $C([0, T]; \mathcal{P}_1(H_{-1}))$ as follows:

$$m \rightsquigarrow v^{(m)} \text{ solution of HJB} \rightsquigarrow \Psi(m) \text{ solution of FP.}$$

To apply Tikhonov's fixed point theorem, we need:

- $\Psi : C \rightarrow C$ for some $C \subset C([0, T]; \mathcal{M}(H_{-1}))$ convex and compact;
- Ψ continuous.

Uniqueness: Approximate solutions and follow classical argument based on Lasry–Lions type monotonicity and separability of the Hamiltonian.

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