On Mean Field Games and Mean Field Control in Infinite Dimensions

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Controlled Interacting Particle Systems

State: For $i = 1, \dots, N$, consider

$$\begin{cases} \mathrm{d}X^i(s) = b(X^i(s), \alpha^i(s), m_{\mathbf{X}(s)})\mathrm{d}s + \sigma(X^i(s), \alpha^i(s), m_{\mathbf{X}(s)})\mathrm{d}W^i(s) \\ \qquad\qquad\qquad +\sigma^0(X^i(s), \alpha^i(s), m_{\mathbf{X}(s)})\mathrm{d}B(s) \\ X^i(t) = x^i \quad \text{i.i.d.} \end{cases}$$

where

- $m_{\mathbf{X}(s)} := \frac{1}{N}\sum_{i=1}^N \delta_{X^i(s)}$ empirical measure
- $(W^i(s))_{s\in[t,T]}$ idiosyncratic noise
- $(B(s))_{s\in[t,T]}$ common noise
- $\alpha^i(\cdot)$ control

Cost Functional:

$$J^i(\alpha^1(\cdot), \dots, \alpha^N(\cdot)) = \mathbb{E}\left[\int_t^T f(X^i(s), \alpha^i(s), m_{\mathbf{X}(s)})\mathrm{d}s + g(X^i(T), m_{\mathbf{X}(T)})\right]$$

where f and g running and terminal cost.

Mean Field Control

Minimize **combined cost** for all particles:

$$\bar{J}(\alpha^1(\cdot), \dots, \alpha^N(\cdot)) = \frac{1}{N} \sum_{i=1}^{N} J^i(\alpha^1(\cdot), \dots, \alpha^N(\cdot))$$

The formal mean field limit $N \to \infty$ is

$$\begin{cases} \mathrm{d}X(s) = b(X(s), \alpha(s), m(s))\mathrm{d}s + \sigma(X(s), \alpha(s), m(s))\mathrm{d}W(s) \\ \qquad\qquad\qquad +\sigma^0(X(s), \alpha(s), m(s))\mathrm{d}B(s) \\ X(t) = x \end{cases}$$

where $m(s) = \mathcal{L}(X(s)|B)$, and cost

$$J(\alpha(\cdot)) = \mathbb{E}\left[\int_t^T f(X(s), \alpha(s), m(s))\mathrm{d}s + g(X(T), m(T))\right].$$

Mean Field Games

Player i minimizes **personal cost** J^i. Search for Nash equilibria:

$$J^i(\alpha^{1,*}, \dots, \alpha^{N,*}) \leq J^i(\alpha^{1,*}, \dots, \alpha^{i-1,*}, \alpha^i, \alpha^{i+1,*}, \dots \alpha^{N,*}) \quad \forall \alpha^i.$$

Limit $N \to \infty$: Given flow $(m(s))_{s\in[t,T]}$, representative player is governed by

$$\begin{cases} \mathrm{d}X(s) = b(X(s), \alpha(s), m(s))\mathrm{d}s + \sigma(X(s), \alpha(s), m(s))\mathrm{d}W(s) \\ \qquad\qquad\qquad +\sigma^0(X(s), \alpha(s), m(s))\mathrm{d}B(s) \\ X(t) = x, \end{cases}$$

and minimizes

$$J^m(\alpha(\cdot)) = \mathbb{E}\left[\int_t^T f(X(s), \alpha(s), m(s))\mathrm{d}s + g(X(T), m(T))\right].$$

Nash equilibrium reduces to fixed-point condition:

$$m(s) = \mathcal{L}(X(s)|B) \quad \forall s \in [t, T].$$

Why MFGs and MFC in Infinite Dimensional Spaces?

Applications:

- Heath–Jarrow–Morton models — *State: Forward rate curve*
- Biological neural networks — *State: Spatial voltage profile*
- Filtering equations — *State: Conditional probability density*
- Economic models with memory — *State: Trajectory of the process*
- ...

Examples of Stochastic Dynamics:

- Stochastic partial differential equations (SPDEs)
- Stochastic delay differential equations (SDDEs)
- Stochastic Volterra Integral Equations (SVIEs)
- SDEs driven by fractional noise
- ...

Natural state is often a function, trajectory, or probability distribution.

Path Dependent Models – An Example

Example: Consider Stochastic Delay Differential Equation

$$\begin{cases} \mathrm{d}y(s) = \tilde{b}\left(y(s), \int_{-d}^{0} \eta(\theta) y(s+\theta)\mathrm{d}\theta, \alpha(s)\right)\mathrm{d}s + \tilde{\sigma}(y(s))\mathrm{d}W(s) \\ y(t) = x^0, \quad y(\theta) = x^1(\theta), \quad \theta \in [-d, 0). \end{cases} \tag{SDDE}$$

Not Markovian! To regain Markovianity, introduce

$$X(s) = (y(s), y(s+\cdot)|_{[-d,0]}) \quad \forall s \geq t$$

$$H = \underbrace{\mathbb{R}}_{\text{Present}} \times \underbrace{L^2(-d, 0)}_{\text{Past}}$$

Then, we can lift (SDDE) to

$$\mathrm{d}X(s) = [AX(s) + b(X(s), \alpha(s))]\mathrm{d}s + \sigma(X(s))\mathrm{d}W(s), \quad X(t) = x.$$

where $A : \mathcal{D}(A) \subset H \to H$,

$$Ax = \begin{bmatrix} 0 \\ x_1' \end{bmatrix}, \quad \mathcal{D}(A) = \left\{x = (x_0, x_1) \in H : \ \ x_1 \in W^{1,2}(-d, 0), \ \ x_1(0) = x_0\right\}$$

MFGs in Hilbert Spaces – Setting

Given flow $(m(s))_{s\in[t,T]}$, representative player is governed by

$$\mathrm{d}X(s) = [AX(s) + b(X(s), \alpha(s), m(s))]\mathrm{d}s + \sigma(X(s))\mathrm{d}W(s), \quad X(t) = x \in H.$$

and minimizes

$$J^m(t, x; \alpha(\cdot)) = \mathbb{E}\left[\int_t^T f(X(s), \alpha(s), m(s))\mathrm{d}s + g(X(T), m(T))\right].$$

Here

- $A : \mathcal{D}(A) \subset H \to H$ closed, densely defined, maximally dissipative
- $(W(s))_{s\in[t,T]}$ cylindrical Wiener process – idiosyncratic noise
- $\sigma : H \to L_2(\Xi, H)$ possibly degenerate
- No common noise $(B(s))_{s\in[t,T]}$

Dynamic Programming: Define value function $v^m : [0, T] \times H \to \mathbb{R}$ as

$$v^m(t, x) := \inf_{\alpha(\cdot)} \mathbb{E}\left[\int_t^T f(X(s), \alpha(s), m(s))\mathrm{d}s + g(X(T), m(T))\right].$$

The MFG System

Value function v^m satisfies Hamilton–Jacobi–Bellman (HJB) equation

$$\partial_t v(t,x) + Lv(t,x) - \mathcal{H}(x, Dv(t,x), m(t)) = 0, \quad v(T,\cdot) = g(\cdot, m(T)) \qquad \text{(HJB)}$$

where

- $L\phi(x) := \langle Ax, D\phi(x)\rangle + \frac{1}{2}\mathrm{Tr}[\sigma(x)\sigma^*(x)D^2\phi(x)]$
- $\mathcal{H}(x,p,m) = \sup_{\alpha\in\Lambda_0}\{-\langle b(x,\alpha,m), p\rangle - f(x,\alpha,m)\}$

Can be used to compute optimal drift in state equation:

$$\mathrm{d}X(t) = [AX(t) - \mathcal{H}_p(X(t), Dv(t,X(t)), m(t))]\mathrm{d}t + \sigma(X(t))\mathrm{d}W(t).$$

Nash equilibrium leads to coupled forward backward system

$$\begin{cases} \partial_t v(t,x) + Lv(t,x) - \mathcal{H}(x, Dv(t,x), m(t)) = 0 \\ \partial_t m(t) - L^* m(t) - \mathrm{div}\,(\mathcal{H}_p(x, Dv(t,x), m(t))m(t)) = 0 \\ v(T,\cdot) = g(\cdot, m(T)), \quad m(0) = m_0 \in \mathcal{P}_1(H) \end{cases} \qquad \text{(MFG System)}$$

Goal: Prove existence and uniqueness of solutions to (MFG System)

MFGs in Hilbert Spaces: Challenges

Consider

$$\begin{cases} \partial_t v(t,x) + Lv(t,x) - \mathcal{H}(x, Dv(t,x), m(t)) = 0 \\ \partial_t m(t) - L^* m(t) - \text{div}\left(\mathcal{H}_p(x, Dv(t,x), m(t)) m(t)\right) = 0 \\ v(T,\cdot) = g(\cdot, m(T)), \quad m(0) = m_0 \in \mathcal{P}_1(H) \end{cases} \qquad \text{(MFG System)}$$

where

- $L\phi(x) := \langle Ax, D\phi(x)\rangle + \frac{1}{2}\text{Tr}[\sigma(x)\sigma^*(x)D^2\phi(x)]$
- $\mathcal{H}(x,p,m) = \sup_{\alpha\in\Lambda_0}\{-\langle b(x,\alpha,m), p\rangle - f(x,\alpha,m)\}$

Challenges:

⚠	H is ∞-dim	$\Longrightarrow$	No local compactness, no Lebesgue measure
⚠	A unbounded	$\Longrightarrow$	Ax only defined for $x \in \mathcal{D}(A)$
⚠	Degenerate noise	$\Longrightarrow$	No smoothing in HJB and FP equations

MFGs in Hilbert Spaces: Related Work

Studying Master Equation:

- (2018) Fouque/Zhang — Linear-Quadratic Model
- (2025+) Ghilli/Ricciardi — Linear-Quadratic Model
- (2025+) Firoozi/Kratsios/Yang — Linear-Quadratic Model

Studying MFG system:

- (2025) Federico/Ghilli/Gozzi — Linear-Quadratic Model
- (2025) Liu/Firoozi — Linear-Quadratic Model
- (2026) Federico/Gozzi/Święch
- (2026+) Liu/Firoozi — Linear-Quadratic Model

In [(2026) Federico/Gozzi/Święch]:

- non-degenerate additive noise $\rightsquigarrow$ framework of mild solutions

Our Contribution:

Existence and Uniqueness of Solution to (MFG System) with Degenerate Noise

MFG System: Notion of Solution

Consider

$$\begin{cases} \partial_t v(t,x) + Lv(t,x) - \mathcal{H}(x, Dv(t,x), m(t)) = 0 \\ \partial_t m(t) - L^* m(t) - \text{div}\left(\mathcal{H}_p(x, Dv(t,x), m(t)) m(t)\right) = 0 \\ v(T,\cdot) = g(\cdot, m(T)), \quad m(0) = m_0 \in \mathcal{P}_1(H) \end{cases} \qquad \text{(MFG System)}$$

Definition

A pair (v, m) is a solution of (MFG system) if

- *v is viscosity solution of HJB satisfying $v(t,\cdot) \in C_b^{1,1}(H_{-1})\ \forall t \in [0,T]$;*
- *m is weak solution of linear FP equation*

$$\partial_t m(t) - L^* m(t) - \text{div}\left(w(t,x) m(t)\right) = 0, \quad m(0) = m_0.$$

with $w(t,x) = \mathcal{H}_p(x, Dv(t,x), m(t))$.

Remark: H_{-1} is weaker Hilbert space topology adapted to unbounded operator A.

B-Continuous Viscosity Solutions

Consider the PDE

$$\partial_t v + \langle Ax, Dv \rangle + \frac{1}{2}\mathrm{Tr}[\sigma(x)\sigma^*(x)D^2 v] - \mathcal{H}(x, Dv, m(t)) = 0$$

Problem: $\langle Ax, D\varphi \rangle$ not well-defined.
Let $B \in L(H)$ be strictly positive, self-adjoint such that $A^*B \in L(H)$ and

$$-A^*B + c_0 B \geq 0, \quad \text{for some } c_0 \geq 0.$$

Work in the space H_{-1} = completion of H w.r.t. $\|x\|_{-1}^2 := \langle Bx, x \rangle$.

Unbounded A: Test with φ such that $D\varphi \in \mathcal{D}(A^*)$ and rewrite

$$\langle Ax, D\varphi \rangle = \langle x, A^* D\varphi \rangle.$$

Compactness: We assume that

$$\boxed{B \text{ is compact}}$$

$C^{1,1}$ Regularity

Theorem (Lasry, Lions (1986))

$v : H \to \mathbb{R}$ semiconcave, semiconvex, continuous $\implies v \in C^{1,1}(H)$.

Value function:

$$v(t,x) := \inf_{\alpha(\cdot)\in\mathcal{A}_t} \mathbb{E}\left[\int_t^T f(X(s),\alpha(s),m(s))\mathrm{d}s + g(X(T),m(T))\right]$$

subject to

$$\mathrm{d}X(s) = [AX(s) + b(X(s),\alpha(s),m(s))]\mathrm{d}s + \sigma(X(s))\mathrm{d}W(s), \quad X(t) = x \in H.$$

Theorem (de Feo, Święch, Wessels (2025))

- *v is semiconcave under suitable assumptions on the coefficients.*
- *v is semiconvex in each of the following two cases:*
  1. *Affine linear state equation + convex cost.*
  2. *$f(x,\alpha,m) = f_0(x,m) + \nu|\alpha|^2$, ν large (or T small)*

Weak Solutions for Fokker–Planck Equations

Linear FP equation:

$$\partial_t m(t) - L^* m(t) - \operatorname{div}\left(w(t,x)m(t)\right) = 0, \quad m(0) = m_0. \tag{1}$$

Definition

A flow of measures $(m(s))_{s\in[0,T]}$ *is a weak solution of* (1) *with initial condition* $m_0 \in \mathcal{P}_1(H)$ *if*

$$\begin{aligned}
&\int_H \varphi(t,x)m(t,\mathrm{d}x) - \int_H \varphi(0,x)m_0(\mathrm{d}x) \\
&= \int_0^t \left(\int_H \left[\partial_t \varphi(s,x) + L\varphi(s,x) - \langle w(s,x), D\varphi(s,x)\rangle\right] m(s,\mathrm{d}x) \right) \mathrm{d}s
\end{aligned}$$

for every $t \in [0,T]$ *and for all test functions* φ.

Uniqueness of Weak Solutions – Outline

Let m_1 and m_2 be two solutions of the linear FP equation. Then

$$\int_H \varphi(T,x)(m_1 - m_2)(T, \mathrm{d}x) \tag{2}$$
$$= \int_0^T \left(\int_H \left[\partial_t \varphi(s,x) + L\varphi(s,x) - \langle w(s,x), D\varphi(s,x) \rangle \right] (m_1 - m_2)(s, \mathrm{d}x) \right) \mathrm{d}s.$$

Consider the adjoint equation

$$\partial_t u + Lu - \langle w(t,x), Du \rangle = f(t,x), \quad u(T,\cdot) = 0.$$

Assume we can solve this equation for any f, and u is smooth.

Testing (2) with $\varphi = u$ yields

$$0 = \int_0^T \left(\int_H f(s,x)(m_1 - m_2)(s, \mathrm{d}x) \right) \mathrm{d}s$$

for all f, hence $m_1 = m_2$.

Weak Solutions – Infinite Dimensional Case

Class of test functions:

$$\mathcal{D}_T = \left\{ \varphi \in C_b^{1,2}([0,T] \times H) : A^* D\varphi(t,x) \in C_b([0,T] \times H; H) \right\}.$$

Adjoint equation:

$$\partial_t u + \langle Ax, Du \rangle + \frac{1}{2} \mathsf{Tr}[\sigma(x)\sigma^*(x) D^2 u] - \langle w(t,x), Du \rangle = f(t,x), \quad u(T,\cdot) = 0.$$

Problem: We cannot solve this equation such that $u \in \mathcal{D}_T$.

Workaround: Use approximation:

1. Approximate w and f so that we can obtain viscosity solution u that is $C^{1,1}$.
2. Approximate u by $u_n \in \mathcal{D}_T$
   - Yosida approximation: $A \rightsquigarrow A_n$
   - Finite dimensional projection P_N in Hilbert space H
   - Mollification on $\mathbb{R}^N$
3. Perform classical argument with u_n and pass to the limit.

MFG System: Well-Posedness

Definition

A pair (v, m) is a solution of the MFG system (HJB)–(FP) if

- *v is viscosity solution of (HJB) satisfying $v(t, \cdot) \in C_b^{1,1}(H_{-1})$ $\forall t \in [0, T]$;*
- *m is solution of linear FP equation with $w(t, x) = \mathcal{H}_p(x, Dv(t, x), m(t))$.*

Theorem (Święch, Wessels (2026+))

Under suitable regularity, Lasry–Lions monotonicity, and separability assumptions, there exists a unique solution (v, m) to the MFG system.

Existence: Define mapping Ψ on a subset of $C([0, T]; \mathcal{P}_1(H_{-1}))$ as follows:

$$m \rightsquigarrow v^{(m)} \text{ solution of HJB} \rightsquigarrow \Psi(m) \text{ solution of FP.}$$

To apply Tikhonov's fixed point theorem, we need:

- $\Psi : C \to C$ for some $C \subset C([0, T]; \mathcal{M}(H_{-1}))$ convex and compact;
- Ψ continuous.

Uniqueness: Lasry–Lions Monotonicity Argument

Assumption

- *Separable Hamiltonian:* $\mathcal{H}(x,p,m) = \mathcal{H}^0(x,p) - F(x,m)$
- *Strict monotonicity of* F*: For all* $m_1, m_2 \in \mathcal{P}_1(H)$, $m_1 \neq m_2$,

$$\int_H \big(F(x,m_1) - F(x,m_2)\big)(m_1 - m_2)(\mathrm{d}x) > 0$$

- *Monotonicity of* g
- *Convexity of* $p \mapsto \mathcal{H}^0(x,p)$

Formal Duality Argument: For two solutions (v_1, m_1), (v_2, m_2), compute $\frac{\mathrm{d}}{\mathrm{d}t}\int_H (v_1 - v_2)\mathrm{d}(m_1 - m_2)$, integrate over $[0,T]$, and drop terms involving g, $\mathcal{H}^0$:

$$\int_0^T \int_H \big(F(x,m_1(t)) - F(x,m_2(t))\big)(m_1(t) - m_2(t))(\mathrm{d}x)\mathrm{d}t \leq 0$$

which implies $m_1 = m_2$, and then $v_1 = v_2$.

Infinite-Dimensional Caveat: $v_1, v_2 \notin \mathcal{D}_T \rightsquigarrow$ use 3-step approximation.

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