Uniqueness by Common Noise for 1D Mean Field Games

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Symmetric N-Player Games

State equation: Players modeled by

$$\mathrm{d}X_s^i = -\alpha_s^i \mathrm{d}s + \gamma \mathrm{d}W_s^i, \quad X_t^i = x, \quad i = 1, \dots, N$$

where $\alpha^i : [t, T] \times \Omega \to \mathbb{R}^d$ denotes the control.

Cost Functional: Minimize

$$J^i(\alpha^1(\cdot), \dots, \alpha^N(\cdot)) = \mathbb{E}\left[\frac{1}{2}\int_t^T |\alpha_s^i|^2 \mathrm{d}s + \mathcal{G}(\mu_{\mathbf{X}_T}, X_T^i)\right]$$

where $\mu_{\mathbf{X}_s} := \frac{1}{N}\sum_{j=1}^N \delta_{X_s^j}$.

Nash Equilibrium: Controls $\alpha^{1,*}(\cdot), \dots, \alpha^{N,*}(\cdot)$ are Nash equilibrium, if

$$J^i(\alpha^{1,*}(\cdot), \dots, \alpha^{i,*}(\cdot), \dots, \alpha^{N,*}(\cdot)) \leq J^i(\alpha^{1,*}(\cdot), \dots, \alpha^i(\cdot), \dots, \alpha^{N,*}(\cdot))$$

for all controls $\alpha^i(\cdot)$, and for all $i \in \{1, \dots, N\}$.

Mean Field Games

State equation: Consider

$$\mathrm{d}X_s = -\alpha_s \mathrm{d}s + \gamma \mathrm{d}W_s, \quad X_t = x.$$

where $\alpha(\cdot)$ denotes the control.
Cost functional: Given a flow of measures $(\mu_s)_{s\in[t,T]}$, minimize

$$J(t,x;\alpha(\cdot)) = \mathbb{E}\left[\frac{1}{2}\int_t^T |\alpha_s|^2 \mathrm{d}s + \mathcal{G}(\mu_T, X_T)\right]$$

Dynamic Programming: Value function $u : [0,T] \times \mathbb{R}^d \to \mathbb{R}$

$$u(t,x) := \inf_{\alpha(\cdot)} J(t,x;\alpha(\cdot))$$

Dynamic Programming Principle: For any $t \le t+h \le T$,

$$u(t,x) = \inf_{\alpha(\cdot)} \mathbb{E}\left[\frac{1}{2}\int_t^{t+h} |\alpha_s|^2 \mathrm{d}s + u(t+h, X_{t+h})\right]$$

MFG System

Hamilton–Jacobi–Bellman Equation:

$$\partial_t u + \tfrac{1}{2}\gamma^2 \operatorname{Tr}[D^2 u] - \tfrac{1}{2}|Du|^2 = 0, \quad u(T,\cdot) = \mathcal{G}(\mu_T,\cdot) \tag{HJB}$$

Optimal feedback control $\alpha^*(t,x) = Du(t,x)$ leads to optimal state:

$$\mathrm{d}X_s = -Du(s,X_s)\mathrm{d}s + \gamma \mathrm{d}W_s.$$

Fokker–Planck Equation:

$$\partial_t \mu - \tfrac{1}{2}\gamma^2 \operatorname{Tr}[D^2 \mu] - \operatorname{div}(\mu Du) = 0, \quad \mu_0 = \mu. \tag{FP}$$

MFG System: Nash equilibrium leads to forward-backward system:

$$\begin{cases} \partial_t \mu - \tfrac{1}{2}\gamma^2 \operatorname{Tr}[D^2 \mu] - \operatorname{div}(\mu Du) = 0 \\ \partial_t u + \tfrac{1}{2}\gamma^2 \operatorname{Tr}[D^2 u] - \tfrac{1}{2}|Du|^2 = 0 \\ \mu_0 = \mu, \quad u(T,\cdot) = \mathcal{G}(\mu_T,\cdot) \end{cases} \tag{MFG System}$$

Forward-Backward Systems and Burgers' Equation

Consider

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}s} x_s = -y_s \\ \frac{\mathrm{d}}{\mathrm{d}s} y_s = 0 \\ x_t = x, \quad y_T = G(x_T) \end{cases}$$

Assume that $y_s = v(s, x_s)$. Then

$$0 = \frac{\mathrm{d}}{\mathrm{d}s} y_s = \partial_s v(s, x_s) + \partial_x v(s, x_s) \frac{\mathrm{d}}{\mathrm{d}s} x_s = \partial_s v(s, x_s) - \partial_x v(s, x_s) v(s, x_s)$$

If this holds true for all initial conditions (t, x), then

$$\partial_t v(t, x) - v(t, x) \partial_x v(t, x) = 0, \quad v(T, x) = G(x).$$

⚠ Develops shocks in finite time $\rightsquigarrow$ non-uniqueness.

Ways to restore uniqueness:

1. Monotonicity: G non-decreasing (in MFG: Lasry–Lions monotonicity)
2. Diffusivity: Add noise to the system

Forward-Backward Systems and Burgers' Equation

Consider

$$\begin{cases} \mathrm{d}X_s = -Y_s \mathrm{d}s + \mathrm{d}W_s \\ \mathrm{d}Y_s = Z_s \mathrm{d}W_s \\ X_t = x, \quad Y_T = G(X_T) \end{cases}$$

Assume that $Y_s = v(s, X_s)$, $Z_s = \partial_x v(s, X_s)$. Then

$$\partial_x v(s, X_s)\mathrm{d}W_s = [\partial_s v(s, X_s) - v(s, X_s)\partial_x v(s, X_s) + \tfrac{1}{2}\partial_x^2 v(s, X_s)]\mathrm{d}s + \partial_x v(s, X_s)\mathrm{d}W_s$$

If this holds true for all initial conditions (t, x), then

$$\partial_t v(t, x) - v(t, x)\partial_x v(t, x) + \tfrac{1}{2}\partial_x^2 v(t, x) = 0, \quad v(T, x) = G(x).$$

⚠ Does not develop shocks $\rightsquigarrow$ uniqueness.

Ways to restore uniqueness:

1. Monotonicity: G non-decreasing (in MFG: Lasry–Lions monotonicity)
2. Diffusivity: Add noise to the system

Common Noise

Our forward-backward system:

$$\begin{cases} \partial_t \mu - \frac{1}{2}\gamma^2 \operatorname{Tr}[D^2 \mu] - \operatorname{div}(\mu Du) = 0 \\ \partial_t u + \frac{1}{2}\gamma^2 \operatorname{Tr}[D^2 u] - \frac{1}{2}|Du|^2 = 0 \\ \mu_0 = \mu, \quad u(T,\cdot) = \mathcal{G}(\mu_T, \cdot) \end{cases} \tag{MFG System}$$

Underlying state space: $\mathcal{P}(\mathbb{T}^d) \rightsquigarrow$ need an infinite dimensional noise:

$$\mathrm{d}X_s = -Du_s(X_s)\mathrm{d}s + \gamma \mathrm{d}W_s + \sum_{k \in \mathbb{Z}^d} \sigma^k(X_s)\mathrm{d}B_s^k,$$

where $\sigma^k = (1+|k|^2)^{-\theta/2} e_k$, $\theta > 0$, and $(e_k)_{k\in\mathbb{Z}^d}$ is Fourier basis on $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$.
Corresponding Fokker–Planck equation for $\mu_t = \mathcal{L}(X_t|\mathbf{B})$:

$$\mathrm{d}\mu_t = \left[\tfrac{1}{2}(\gamma^2 + \gamma_0^2) \operatorname{Tr}[D^2 \mu_t] + \operatorname{div}(\mu_t Du_t)\right] \mathrm{d}t - \sum_{k\in\mathbb{Z}^d} \operatorname{div}\left(\sigma^k \mu_t\right) \mathrm{d}B_t^k.$$

Terminal condition for HJB equation:

$$u_T(\cdot) = \mathcal{G}(\mu_T, \cdot)$$

which is random $\rightsquigarrow$ Need backward SPDE for value function u!

MFG System with Common Noise

Given random terminal condition $\xi \in L^2_{\mathcal{F}_T}(\Omega)$, consider simple BSDE

$$\mathrm{d}Y_t = 0, \quad Y_T = \xi$$

⚠ Adapted solution doesn't exist! Natural candidate: $Y_t = \mathbb{E}[\xi|\mathcal{F}_t]$.

$$Y_t = \mathbb{E}[\xi|\mathcal{F}_t] = \mathbb{E}[\xi] + \int_0^t Z_s \mathrm{d}W_s \iff \boxed{\mathrm{d}Y_t = Z_t \mathrm{d}W_t, \quad Y_T = \xi}$$

MFG system:

$$\begin{cases} \mathrm{d}\mu_t = \left[\frac{1}{2}(\gamma^2 + \gamma_0^2)\,\mathrm{Tr}[D^2\mu_t] + \mathrm{div}\,(\mu_t Du_t)\right]\mathrm{d}t - \sum_{k\in\mathbb{Z}^d} \mathrm{div}\left(\sigma^k \mu_t\right)\mathrm{d}B_t^k \\ \mathrm{d}u_t = -\left[\frac{1}{2}(\gamma^2 + \gamma_0^2)\,\mathrm{Tr}[D^2 u_t] - \frac{1}{2}|Du_t|^2 + \sum_{k\in\mathbb{Z}^d} \sigma^k Dv_t^k\right]\mathrm{d}t + \sum_{k\in\mathbb{Z}^d} v_t^k \mathrm{d}B_t^k \\ \mu_0 = \mu, \quad u_T(x) = \mathcal{G}(\mu_T, x). \end{cases}$$

Main result: Well-posedness on arbitrary time horizon *without* monotonicity.

Strategy: Decouple the MFG system using the common noise.

Decoupling MFG System via Girsanov's Theorem

Expand

$$Du_t = \sum_{k\in\mathbb{Z}^d} \langle Du_t, e_k\rangle_{L^2(\mathbb{T}^d)} e_k = \sum_{k\in\mathbb{Z}^d} \langle Du_t, e_k\rangle_{L^2(\mathbb{T}^d)} (1+|k|^2)^{\frac{\theta}{2}} \sigma^k$$

Thus,

$$Du_t(x)\mathrm{d}t - \sum_{k\in\mathbb{Z}^d} \sigma^k(x)\mathrm{d}B_t^k = \sum_{k\in\mathbb{Z}^d} \sigma^k(x)\left[(1+|k|^2)^{\frac{\theta}{2}} \langle Du_t, e_k\rangle \mathrm{d}t - \mathrm{d}B_t^k\right]$$

Need to control

$$\sum_{k\in\mathbb{Z}^d} \int_0^t (1+|k|^2)^{\frac{\theta}{2}} \langle Du_s, e_k\rangle \mathrm{d}B_s^k.$$

By BMO condition, it suffices to prove a bound on

$$\sup_\tau \left\| \mathbb{E}\left[\sum_{k\in\mathbb{Z}^d} \int_\tau^T (1+|k|^2)^\theta |\langle Du_t, e_k\rangle|^2 \mathrm{d}t \middle| \mathcal{F}_\tau \right] \right\|_{L^\infty(\Omega)}$$

$$= \sup_\tau \left\| \mathbb{E}\left[\int_\tau^T \|Du_t\|^2_{H^\theta(\mathbb{T}^d)} \mathrm{d}t \middle| \mathcal{F}_\tau \right] \right\|_{L^\infty(\Omega)}$$

Regularity for Backward Stochastic HJB Equations

Consider

$$\begin{cases} \mathrm{d}u_t = -\left[\frac{1}{2}(\gamma^2+\gamma_0^2)\,\mathrm{Tr}[D^2u_t] - \frac{1}{2}|Du_t|^2 + \sum_{k\in\mathbb{Z}^d}\sigma^k Dv_t^k\right]\mathrm{d}t + \sum_{k\in\mathbb{Z}^d} v_t^k \mathrm{d}B_t^k \\ u_T(x) = \mathcal{G}(\mu_T, x). \end{cases}$$

Let us consider the linear equation (i.e. without $|Du_t|^2$). We have

$$\begin{aligned} \mathrm{d}\|D^{\ell+1}u_t\|^2_{L^2(\mathbb{T}^d)} = \Big[(\gamma^2+\gamma_0^2)\|D^{\ell+2}u_t\|^2_{L^2(\mathbb{T}^d)} + 2\Big\langle D^{\ell+2}u_t, D^\ell\Big(\sum_{k\in\mathbb{Z}^d}\sigma^k Dv_t^k\Big)\Big\rangle\Big]\mathrm{d}t \\ + 2\sum_{k\in\mathbb{Z}^d}\langle D^{\ell+1}u_t, D^{\ell+1}v_t^k\rangle\,\mathrm{d}B_t^k + \sum_{k\in\mathbb{Z}^d}\|D^{\ell+1}v_t^k\|^2_{L^2(\mathbb{T}^d)}\mathrm{d}t \end{aligned}$$

A few remarks:

1. $\sum_{k\in\mathbb{Z}^d} D^\ell\sigma^k Dv_t^k \sim \sum_{k\in\mathbb{Z}^d} k^\ell(1+|k|^2)^{-\frac{\theta}{2}} e_k Dv_t^k$ requires $\ell + d/2 < \theta$
2. Slide before: $\mathbb{E}[\int_\tau^T \|Du_t\|^2_{H^\theta(\mathbb{T}^d)}\mathrm{d}t|\mathcal{F}_\tau] < \infty \implies \theta < \ell + 1$

Thus, we need: $d = 1$

The Decoupled System

Applying Girsanov to the MFG system yields

$$
\begin{cases}
\mathrm{d}\mu_t = \frac{1}{2}(\gamma^2+\gamma_0^2)D^2\mu_t \mathrm{d}t - D\left(\mu_t \sum_{k\in\mathbb{Z}} \sigma^k \mathrm{d}\tilde{B}_t^k\right) \\
\mathrm{d}u_t = -\left[\frac{1}{2}(\gamma^2+\gamma_0^2)D^2u_t - \frac{1}{2}|Du_t|^2 + \sum_{k\in\mathbb{Z}} \sigma^k(x) Dv_t^k\right]\mathrm{d}t \\
\qquad\quad + \sum_{k\in\mathbb{Z}}(1+k^2)^{\frac{\theta}{2}} v_t^k \langle Du_t(\cdot), e_k\rangle \mathrm{d}t + \sum_{k\in\mathbb{Z}} v_t^k \mathrm{d}\tilde{B}_t^k \\
\mu_0 = \mu, \quad u_T(x) = \mathcal{G}(\mu_T, x),
\end{cases}
\tag{$*$}
$$

- Unique solution on short time horizon via original system and Cauchy–Lipschitz theory.
- In order to extend solution to arbitrary time horizon, we need stability:

$$
\|u_t^\mu - u_t^\nu\|_{L^\infty(\mathbb{T})} + \|Du_t^\mu - Du_t^\nu\|_{L^\infty(\mathbb{T})} \leq C\mathbf{d}_2(\mu,\nu),
$$

u^μ, u^ν are solutions of $(*)$ when $\mu_0 = \mu$ and $\mu_0 = \nu$, respectively.

Stability Estimate

We have

$$\begin{aligned}&\mathrm{d}\|D^{\ell+1}\delta u_t\|^2\\&\geq -\left[C+\frac{1}{\varepsilon}\|u_t\|^2_{H^{\theta+1}(\mathbb{T})}+\varepsilon\sum_{k\in\mathbb{Z}}\|D^{\ell+1}v_t^k\|^2_{L^2(\mathbb{T})}\right]\|D^{\ell+1}\delta u_t\|^2_{L^2(\mathbb{T})}\mathrm{d}t+\dots\end{aligned}$$

We need

$$\mathbb{E}\left[\exp\left(\int_0^T \text{red term}\,\mathrm{d}t\right)\right]\leq C$$

BMO provides exponential moment: For $\eta_0>0$ small,

$$\mathbb{E}\left[\exp\left(\eta_0\int_0^T\left(\|u_t\|^2_{H^{\ell+2}(\mathbb{T})}+\sum_{k\in\mathbb{Z}}\|v_t^k\|^2_{H^{\ell+1}(\mathbb{T})}\right)\mathrm{d}t\right)\right]\leq C$$

Since $\theta+1<\ell+2$, we have

$$\|u_t\|^2_{H^{\theta+1}(\mathbb{T})}\leq C(\eta)\|u_t\|^2_{H^{\ell+1}(\mathbb{T})}+\eta\|u_t\|^2_{H^{\ell+2}(\mathbb{T})}$$

for any $\eta>0$. Now, choose ε and η sufficiently small.

Preprint coming soon...