Regularization by Common Noise for Mean Field Games and Mean Field Control

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Mean Field Games

State equation: Consider

$$\mathrm{d}X_s^\alpha = -\alpha_s \mathrm{d}s + \gamma \mathrm{d}W_s, \quad X_t^\alpha = x.$$

where $\alpha(\cdot)$ denotes the control.
Cost functional: Given a flow of measures $(\mu_s)_{s\in[t,T]}$, minimize

$$J(t,x;\alpha(\cdot)) = \mathbb{E}\left[\frac{1}{2}\int_t^T |\alpha_s|^2 \mathrm{d}s + \mathcal{G}(\mu_T, X_T^\alpha)\right]$$

Dynamic Programming: Value function $u : [0,T]\times\mathbb{R}^d \to \mathbb{R}$

$$u(t,x) := \inf_{\alpha(\cdot)} J(t,x;\alpha(\cdot))$$

Dynamic Programming Principle: For any $t \le t+h \le T$,

$$u(t,x) = \inf_{\alpha(\cdot)} \mathbb{E}\left[\frac{1}{2}\int_t^{t+h} |\alpha_s|^2 \mathrm{d}s + u(t+h, X_{t+h}^\alpha)\right]$$

MFG System

Hamilton–Jacobi–Bellman Equation:

$$\partial_t u + \tfrac{1}{2}\gamma^2 \Delta u - \tfrac{1}{2}|Du|^2 = 0, \quad u(T,\cdot) = \mathcal{G}(\mu_T, \cdot) \tag{HJB}$$

Optimal feedback control $\alpha^*(t,x) = Du(t,x)$ leads to optimal state:

$$\mathrm{d}X_s = -Du(s, X_s)\mathrm{d}s + \gamma \mathrm{d}W_s.$$

Fokker–Planck Equation:

$$\partial_t \mu - \tfrac{1}{2}\gamma^2 \Delta \mu - \operatorname{div}(\mu Du) = 0, \quad \mu_0 = \mu. \tag{FP}$$

MFG System: Nash equilibrium leads to forward-backward system:

$$\begin{cases} \partial_t \mu - \tfrac{1}{2}\gamma^2 \Delta \mu - \operatorname{div}(\mu Du) = 0 \\ \partial_t u + \tfrac{1}{2}\gamma^2 \Delta u - \tfrac{1}{2}|Du|^2 = 0 \\ \mu_0 = \mu, \quad u(T,\cdot) = \mathcal{G}(\mu_T, \cdot) \end{cases} \tag{MFG System}$$

Forward-Backward Systems and Burgers' Equation

Consider

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}s} x_s = -y_s \\ \frac{\mathrm{d}}{\mathrm{d}s} y_s = 0 \\ x_t = x, \quad y_T = G(x_T) \end{cases}$$

Assume that $y_s = v(s, x_s)$. Then

$$0 = \frac{\mathrm{d}}{\mathrm{d}s} y_s = \partial_s v(s, x_s) + \partial_x v(s, x_s) \frac{\mathrm{d}}{\mathrm{d}s} x_s = \partial_s v(s, x_s) - \partial_x v(s, x_s) v(s, x_s)$$

If this holds true for all initial conditions (t, x), then

$$\partial_t v(t, x) - v(t, x) \partial_x v(t, x) = 0, \quad v(T, x) = G(x).$$

⚠ Develops shocks in finite time $\rightsquigarrow$ non-uniqueness.

Ways to restore uniqueness:

1. Monotonicity: G non-decreasing (in MFG: Lasry–Lions monotonicity)
2. Diffusivity: Add noise to the system

Forward-Backward Systems and Burgers' Equation

Consider

$$\begin{cases} \mathrm{d}X_s = -Y_s \mathrm{d}s + \mathrm{d}W_s \\ \mathrm{d}Y_s = Z_s \mathrm{d}W_s \\ X_t = x, \quad Y_T = G(X_T) \end{cases}$$

Assume that $Y_s = v(s, X_s)$, $Z_s = \partial_x v(s, X_s)$. Then

$$\partial_x v(s, X_s)\mathrm{d}W_s = [\partial_s v(s, X_s) - v(s, X_s)\partial_x v(s, X_s) + \tfrac{1}{2}\partial_x^2 v(s, X_s)]\mathrm{d}s + \partial_x v(s, X_s)\mathrm{d}W_s$$

If this holds true for all initial conditions (t, x), then

$$\partial_t v(t, x) - v(t, x)\partial_x v(t, x) + \tfrac{1}{2}\partial_x^2 v(t, x) = 0, \quad v(T, x) = G(x).$$

⚠ Does not develop shocks $\rightsquigarrow$ uniqueness.

Ways to restore uniqueness:

1. Monotonicity: G non-decreasing (in MFG: Lasry–Lions monotonicity)
2. Diffusivity: Add noise to the system

Common Noise

Our forward-backward system:

$$\begin{cases} \partial_t \mu - \frac{1}{2}\gamma^2 \Delta \mu - \operatorname{div}(\mu Du) = 0 \\ \partial_t u + \frac{1}{2}\gamma^2 \Delta u - \frac{1}{2}|Du|^2 = 0 \\ \mu_0 = \mu, \quad u(T,\cdot) = \mathcal{G}(\mu_T, \cdot) \end{cases} \qquad \text{(MFG System)}$$

Underlying state space: $\mathcal{P}(\mathbb{T}^d) \rightsquigarrow$ need an infinite dimensional noise:

$$\mathrm{d}X_s = -Du_s(X_s)\mathrm{d}s + \gamma \mathrm{d}W_s + \sum_{k\in\mathbb{Z}^d} \sigma^k(X_s)\mathrm{d}B_s^k,$$

where

$$\sigma^k(x) = (1+|k|^2)^{-\frac{\theta}{2}} e_k(x), \quad \theta > 0,$$

and $(e_k)_{k\in\mathbb{Z}^d}$ is Fourier basis on $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$.

Fokker–Planck equation for $\mu_t = \mathcal{L}(X_t|\mathcal{F}_t^{\mathbf{B}})$:

$$\mathrm{d}\mu_t = \left[\tfrac{1}{2}(\gamma^2+\gamma_0^2)\Delta\mu_t + \operatorname{div}(\mu_t Du_t)\right]\mathrm{d}t - \sum_{k\in\mathbb{Z}^d} \operatorname{div}\left(\sigma^k \mu_t\right)\mathrm{d}B_t^k.$$

Deriving the Backward Stochastic HJB Equation

Terminal condition for HJB equation is random $\rightsquigarrow$ Need backward SPDE:

$$\mathrm{d}u_t = f_t \mathrm{d}t + \sum_{k\in\mathbb{Z}^d} v_t^k \mathrm{d}B_t^k, \quad u_T = \mathcal{G}(\mu_T, \cdot)$$

Itô–Wentzell formula and dynamic programming principle

$$0 = \mathbb{E}\Bigg[\int_t^{t+h} \Big(f_r(X_r) + \frac{1}{2}(\gamma^2+\gamma_0^2)\Delta u_r(X_r) \\ - \frac{1}{2}|Du_r(X_r)|^2 + \sum_{k\in\mathbb{Z}^d} \sigma^k(X_r) Dv_r^k(X_r)\Big)\mathrm{d}r \,\Big|\, \mathcal{F}_t\Bigg].$$

This yields MFG system:

$$\begin{cases} \mathrm{d}\mu_t = \left[\frac{1}{2}(\gamma^2+\gamma_0^2)\Delta\mu_t + \operatorname{div}(\mu_t Du_t)\right]\mathrm{d}t - \sum_{k\in\mathbb{Z}^d} \operatorname{div}\left(\sigma^k\mu_t\right)\mathrm{d}B_t^k \\ \mathrm{d}u_t = -\left[\frac{1}{2}(\gamma^2+\gamma_0^2)\Delta u_t - \frac{1}{2}|Du_t|^2 + \sum_{k\in\mathbb{Z}^d}\sigma^k Dv_t^k\right]\mathrm{d}t + \sum_{k\in\mathbb{Z}^d} v_t^k \mathrm{d}B_t^k \\ \mu_0 = \mu, \quad u_T(x) = \mathcal{G}(\mu_T, x). \end{cases}$$

Main result: Well-posedness on arbitrary time horizon *without* monotonicity.

Decoupling MFG System via Girsanov's Theorem

Expand

$$Du_t = \sum_{k\in\mathbb{Z}^d} \langle Du_t, e_k\rangle_{L^2(\mathbb{T}^d)} e_k = \sum_{k\in\mathbb{Z}^d} \langle Du_t, e_k\rangle_{L^2(\mathbb{T}^d)} (1+|k|^2)^{\frac{\theta}{2}} \sigma^k$$

Thus,

$$Du_t(x)\mathrm{d}t - \sum_{k\in\mathbb{Z}^d} \sigma^k(x)\mathrm{d}B_t^k = \sum_{k\in\mathbb{Z}^d} \sigma^k(x)\left[(1+|k|^2)^{\frac{\theta}{2}} \langle Du_t, e_k\rangle \mathrm{d}t - \mathrm{d}B_t^k\right]$$

Need to control

$$\sum_{k\in\mathbb{Z}^d} \int_0^t (1+|k|^2)^{\frac{\theta}{2}} \langle Du_s, e_k\rangle \mathrm{d}B_s^k.$$

By BMO condition, it suffices to prove a bound on

$$\sup_\tau \left\| \mathbb{E}\left[\sum_{k\in\mathbb{Z}^d} \int_\tau^T (1+|k|^2)^\theta |\langle Du_t, e_k\rangle|^2 \mathrm{d}t \middle| \mathcal{F}_\tau \right] \right\|_{L^\infty(\Omega)}$$

$$= \sup_\tau \left\| \mathbb{E}\left[\int_\tau^T \|Du_t\|^2_{H^\theta(\mathbb{T}^d)} \mathrm{d}t \middle| \mathcal{F}_\tau \right] \right\|_{L^\infty(\Omega)}$$

Regularity for Linear Backward SPDEs

Consider

$$\begin{cases} \mathrm{d}u_t = -\left[\frac{1}{2}(\gamma^2+\gamma_0^2)\Delta u_t + \sum_{k\in\mathbb{Z}^d} \sigma^k D v_t^k\right]\mathrm{d}t + \sum_{k\in\mathbb{Z}^d} v_t^k \mathrm{d}B_t^k \\ u_T(x) = \mathcal{G}(\mu_T, x). \end{cases}$$

We have

$$\begin{aligned} \mathrm{d}\|D^{\ell+1}u_t\|^2_{L^2(\mathbb{T}^d)} = \Bigg[(\gamma^2+\gamma_0^2)\|D^{\ell+2}u_t\|^2_{L^2(\mathbb{T}^d)} + 2\left\langle D^{\ell+2}u_t, D^\ell\left(\sum_{k\in\mathbb{Z}^d}\sigma^k D v_t^k\right)\right\rangle\Bigg]\mathrm{d}t \\ + 2\sum_{k\in\mathbb{Z}^d}\left\langle D^{\ell+1}u_t, D^{\ell+1}v_t^k\right\rangle \mathrm{d}B_t^k + \sum_{k\in\mathbb{Z}^d}\|D^{\ell+1}v_t^k\|^2_{L^2(\mathbb{T}^d)}\mathrm{d}t \end{aligned}$$

A few remarks:

1. $\sum_{k\in\mathbb{Z}^d} D^\ell\sigma^k D v_t^k \sim \sum_{k\in\mathbb{Z}^d} k^\ell(1+|k|^2)^{-\frac{\theta}{2}} e_k D v_t^k$ requires $\ell + d/2 < \theta$
2. Slide before: $\mathbb{E}[\int_\tau^T \|Du_t\|^2_{H^\theta(\mathbb{T}^d)}\mathrm{d}t|\mathcal{F}_\tau] < \infty \implies \theta \le \ell + 1$

Thus, we need: $d = 1$

Regularity for Backward Stochastic HJB Equations

Lipschitz Nonlinearity:

$$\begin{cases} \mathrm{d}u_t = -\left[\frac{1}{2}(\gamma^2+\gamma_0^2)D^2u_t - f(Du_t) + \sum_{k\in\mathbb{Z}}\sigma^k Dv_t^k\right]\mathrm{d}t + \sum_{k\in\mathbb{Z}} v_t^k \mathrm{d}B_t^k \\ u_T(x) = \mathcal{G}(\mu_T, x). \end{cases}$$

- *Strategy:* Replace $f(Du_t) = f(D\psi_t)$ and use fixed point argument for $\psi \mapsto u^\psi$.
- *Main ingredient:* Bismut–Elworthy formula:

$$\begin{aligned} Du_t^\psi(x) = \overline{\mathbf{E}\times\mathbb{E}}\Big[D_x\mathcal{G}(\mu_T, X_T^{0,t,x})D_xX_T^{0,t,x}\,\Big|\,\mathcal{F}_t\Big] \\ + \frac{1}{\gamma}\int_t^T \overline{\mathbf{E}\times\mathbb{E}}\left[\frac{1}{s-t}f\Big(D\psi_s(X_s^{0,t,x})\Big)\left(\int_t^s D_xX_r^{0,t,x}\mathrm{d}W_r\right)\,\middle|\,\mathcal{F}_t\right]\mathrm{d}s. \end{aligned}$$

Quadratic Nonlinearity:

$$\begin{cases} \mathrm{d}u_t = -\left[\frac{1}{2}(\gamma^2+\gamma_0^2)D^2u_t - \frac{1}{2}|Du_t|^2 + \sum_{k\in\mathbb{Z}}\sigma^k Dv_t^k\right]\mathrm{d}t + \sum_{k\in\mathbb{Z}} v_t^k \mathrm{d}B_t^k \\ u_T(x) = \mathcal{G}(\mu_T, x). \end{cases}$$

Truncate quadratic nonlinearity and derive a bound on $D_xu_t^N$ that is uniform in N.

The Decoupled System

Applying Girsanov to the MFG system yields

$$
\begin{cases}
\mathrm{d}\mu_t = \frac{1}{2}(\gamma^2+\gamma_0^2)D^2\mu_t \mathrm{d}t - D\left(\mu_t \sum_{k\in\mathbb{Z}} \sigma^k \mathrm{d}\tilde{B}_t^k\right) \\
\mathrm{d}u_t = -\left[\frac{1}{2}(\gamma^2+\gamma_0^2)D^2u_t - \frac{1}{2}|Du_t|^2 + \sum_{k\in\mathbb{Z}} \sigma^k(x) Dv_t^k\right]\mathrm{d}t \\
\qquad\quad + \sum_{k\in\mathbb{Z}}(1+k^2)^{\frac{\theta}{2}} v_t^k \langle Du_t(\cdot), e_k\rangle \mathrm{d}t + \sum_{k\in\mathbb{Z}} v_t^k \mathrm{d}\tilde{B}_t^k \\
\mu_0 = \mu, \quad u_T(x) = \mathcal{G}(\mu_T, x),
\end{cases}
\tag{$*$}
$$

- Unique solution on short time horizon via original system and Cauchy–Lipschitz theory.
- In order to extend solution to arbitrary time horizon, we need stability:

$$
\|u_t^\mu - u_t^\nu\|_{L^\infty(\mathbb{T})} + \|Du_t^\mu - Du_t^\nu\|_{L^\infty(\mathbb{T})} \leq C\mathbf{d}_2(\mu,\nu),
$$

u^μ, u^ν are solutions of $(*)$ when $\mu_0 = \mu$ and $\mu_0 = \nu$, respectively.

Stability Estimate

We have

$$\begin{aligned}&\mathrm{d}\|D^{\ell+1}\delta u_t\|^2\\&\geq -\left[C+\frac{1}{\varepsilon}\|u_t\|^2_{H^{\theta+1}(\mathbb{T})}+\varepsilon\sum_{k\in\mathbb{Z}}\|D^{\ell+1}v_t^k\|^2_{L^2(\mathbb{T})}\right]\|D^{\ell+1}\delta u_t\|^2_{L^2(\mathbb{T})}\mathrm{d}t+\dots\end{aligned}$$

We need

$$\mathbb{E}\left[\exp\left(\int_0^T \text{red term}\,\mathrm{d}t\right)\right]\leq C$$

BMO provides exponential moment: For $\eta_0>0$ small,

$$\mathbb{E}\left[\exp\left(\eta_0\int_0^T\left(\|u_t\|^2_{H^{\ell+2}(\mathbb{T})}+\sum_{k\in\mathbb{Z}}\|v_t^k\|^2_{H^{\ell+1}(\mathbb{T})}\right)\mathrm{d}t\right)\right]\leq C$$

Since $\theta+1<\ell+2$, we have

$$\|u_t\|^2_{H^{\theta+1}(\mathbb{T})}\leq C(\eta)\|u_t\|^2_{H^{\ell+1}(\mathbb{T})}+\eta\|u_t\|^2_{H^{\ell+2}(\mathbb{T})}$$

for any $\eta>0$. Now, choose ε and η sufficiently small.

Application to Mean Field Control

Minimize

$$J(s, X; \alpha) = G\left(\mu_T^\alpha\right) + \frac{1}{2}\mathbb{E}\left[\int_s^T |\alpha_t|^2 \mathrm{d}t\right]$$

where $(\mu_t^\alpha)_{t\in[s,T]}$ is the law $\mathcal{L}(X_t^\alpha)$ of

$$\mathrm{d}X_t^\alpha = -\alpha_t \mathrm{d}t + \gamma \mathrm{d}W_t, \quad X_s^\alpha = X.$$

Remark: Even for $\gamma > 0$, the value function is not differentiable in general!

Common noise regularization: Consider

$$\mathcal{V}(s, \mu) := \inf_{\substack{X\in L^2(\Omega)\\ \mathcal{L}(X)=\mu}} \inf_\alpha \mathbb{E}\left[G\left(\mu_T^\alpha\right) + \frac{1}{2}\int_s^T |\alpha_t|^2 \mathrm{d}t\right]$$

where $(\mu_t^\alpha)_{t\in[s,T]}$ is the conditional law $\mathcal{L}(X_t^\alpha|\mathcal{F}_t^{\mathbf{B}})$ of

$$\mathrm{d}X_t^\alpha = -\alpha_t \mathrm{d}t + \gamma \mathrm{d}W_t + \sum_{k\in\mathbb{Z}} \sigma^k(X_t^\alpha)\mathrm{d}B_t^k, \quad X_s^\alpha = X.$$

Main result: $\mathcal{V}$ is Lions differentiable and $D_\mu \mathcal{V}$ is optimal feedback control.

Potential Mean Field Game

Consider the **MFG system**

$$\begin{cases} \mathrm{d}\mu_t = \Big[\frac{1}{2}(\gamma^2+\gamma_0^2)D^2\mu_t + D\big(Du_t\mu_t\big)\Big]\mathrm{d}t - \sum_{k\in\mathbb{Z}} D\big(\sigma^k\mu_t\big)\mathrm{d}B_t^k, \\ \mathrm{d}u_t = -\Big[\frac{1}{2}(\gamma^2+\gamma_0^2)D^2u_t - \frac{1}{2}|Du_t|^2 + \sum_{k\in\mathbb{Z}}\sigma^k(x)Dv_t^k\Big]\mathrm{d}t + \sum_{k\in\mathbb{Z}} v_t^k\mathrm{d}B_t^k, \\ \mu_s = \mu, \qquad u_T(x) = \dfrac{\delta G}{\delta\mu}(\mu_T, x). \end{cases}$$

Connection between MFC and MFG:

$$\frac{\delta\mathcal{V}}{\delta\mu}(s,\mu,x) = u_s(x) \quad \text{and} \quad D_\mu\mathcal{V}(s,\mu,x) = D_x u_s(x)$$

where u_s is solution of MFG system when $\mu_s = \mu$.

Proof outline:

1. Show that Pontryagin maximum principle is necessary and sufficient for optimality in short time.
2. Show that $D_x u_t(X_t)$ satisfies Pontryagin maximum principle.
3. Extend to arbitrary time horizon using results on MFG system.

Preprint coming soon...